Seminario Iberoamericano de
Análisis Matemático y Matemática Aplicada

Polinomios ortogonales Sobolev en varias variables

Omar Salazar-Morales
omsalazarmo@unal.edu.co

Universidad Nacional de Colombia
Departamento de Matemáticas
Grupo de investigación en polinomios ortogonales y aplicaciones
Bogotá, D.C., Colombia

Diciembre, 2021

Contenido

Notación y preliminares

- $\delta_{n,m}$ y $(x)_n$ son la delta de Kronecker delta el símbolo de Pochhammer:

$$\delta_{n,m} := \begin{cases} 1, & n = m, \\ 0, & n \neq m, \end{cases} \quad n, m \in \mathbb{Z},$$

$$(x)_n := \begin{cases} x(x+1)(x+2)\cdots(x+n-1), & n \geq 1, \\ 1, & n = 0, \end{cases} \quad x \in \mathbb{R}, \quad n \in \mathbb{N}_0.$$

- $\Gamma(x)$ es la función gamma
- Sea $d \in \mathbb{N}$. $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}_0^d$, denota un multi-índice para el cual $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_d$.
- $$\binom{x}{n} := \frac{(x-n+1)_n}{n!}, \quad x \in \mathbb{R}, \quad n \in \mathbb{N}_0.$$

Notación y preliminares

- $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$
- Para dos vectores fila o columna $\mathbf{x}$ e $\mathbf{y}$, $\mathbf{x} \cdot \mathbf{y}$ y $\|\mathbf{x}\|$ son el producto punto $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{d} x_i y_i$ y la norma Euclidea $\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$.
- Si α es multi-índice y si $\partial_i^{\alpha_i} := \partial^{\alpha_i} / \partial x_i^{\alpha_i}$ es la α_i-ésima derivada parcial entonces

  $$\partial^\alpha := \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_d^{\alpha_d}$$

  es un operador diferencial de orden $|\alpha|$
- Operadores diferenciales:

$$\triangle := \sum_{j=1}^{d} \partial_j^2, \quad \nabla := \left(\partial_1, \partial_2, \ldots, \partial_d\right)^T, \quad \left\langle \mathbf{x}^T, \nabla \right\rangle := \sum_{j=1}^{d} x_j \partial_j,$$

Notación y preliminares

- $\mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}$ es un monomio
- Un polinomio P en varias variables:

$$P(\mathbf{x}) = \sum_{\alpha} c_\alpha \mathbf{x}^\alpha$$

- El espacio de polinomios es denotado por Π^d
- El espacio de polinomios de grado a lo más n es denotado por Π_n^d

$$\dim \Pi_n^d = \binom{n+d}{n}.$$

- Un polinomio homogéneo P de grado n en varias variables:

$$P(\mathbf{x}) = \sum_{|\alpha|=n} c_\alpha \mathbf{x}^\alpha$$

Notación y preliminares

- El espacio de polinomios homogeneos denotado por $\mathscr{P}_n^d$

$$\mathscr{P}_n^d = \left\{ P \in \Pi_n^d : P(\mathbf{x}) = \sum_{|\alpha|=n} c_\alpha \mathbf{x}^\alpha \right\},$$

$$\dim \mathscr{P}_n^d = \# \left\{ \alpha \in \mathbb{N}_0^d : |\alpha| = n \right\} = r_n^d = \binom{n+d-1}{n}.$$

- El espacio de polinomios armónicos denotado por $\mathscr{H}_n^d$

$$\mathscr{H}_n^d = \left\{ P \in \mathscr{P}_n^d : \triangle P = 0 \right\},$$

$$a_n^d := \dim \mathscr{H}_n^d = \binom{n+d-1}{d-1} - \binom{n+d-3}{d-1}.$$

Un producto interno estándar $\langle \cdot, \cdot \rangle$ se caracteriza porque es simétrico con respecto al operador de multiplicación:

$$\langle x_i P, Q \rangle = \langle P, x_i Q \rangle \,, \quad P, Q \in \Pi^d, \quad 1 \leq i \leq d, \tag{1}$$

Esta propiedad implica que, para una base $\{P^n_\alpha : |\alpha| = n\}$ de

$$\mathscr{V}_n^d = \left\{ P \in \Pi_n^d : \langle P, Q \rangle = 0, \forall Q \in \Pi_{n-1}^d \right\}, \quad \dim \mathscr{V}_n^d = r_n^d = \binom{n+d-1}{n},$$

escrita como vector columna:

$$\mathbb{P}_n(\mathbf{x}) := \left(P^n_{\alpha^{(1)}}(\mathbf{x}), P^n_{\alpha^{(2)}}(\mathbf{x}), \ldots, P^n_{\alpha^{(r_n^d)}}(\mathbf{x}) \right)^T,$$

ordenados con orden lexicográfico inverso, se tienen:

Polinomios Sobolev vs. Polinomios estándar

- Una relación a tres términos:

$$x_i \mathbb{P}_n(\mathbf{x}) = \mathbf{A}_{n,i}\mathbb{P}_{n+1}(\mathbf{x}) + \mathbf{B}_{n,i}\mathbb{P}_n(\mathbf{x}) + \mathbf{C}_{n,i}\mathbb{P}_{n-1}(\mathbf{x}),$$
$$n \geq 0, \quad 1 \leq i \leq d, \quad \mathbb{P}_{-1} := 0.$$

- La fórmula de Christoffel-Darboux:

$$K_n(\mathbf{x},\mathbf{y}) = \sum_{k=0}^{n} \mathbb{P}_k^T(\mathbf{x})(\mathbf{H}_k)^{-1}\mathbb{P}_k(\mathbf{y})$$
$$= \begin{cases} \dfrac{[\mathbf{A}_{n,i}\mathbb{P}_{n+1}(\mathbf{x})]^T(\mathbf{H}_n)^{-1}\mathbb{P}_n(\mathbf{y}) - \mathbb{P}_n^T(\mathbf{x})(\mathbf{H}_n)^{-1}[\mathbf{A}_{n,i}\mathbb{P}_{n+1}(\mathbf{y})]}{x_i - y_i}, & \mathbf{x} \neq \mathbf{y}, \\ \mathbb{P}_n^T(\mathbf{x})(\mathbf{H}_n)^{-1}[\mathbf{A}_{n,i}\partial_i\mathbb{P}_{n+1}(\mathbf{x})] - [\mathbf{A}_{n,i}\mathbb{P}_{n+1}(\mathbf{x})]^T(\mathbf{H}_n)^{-1}\partial_i\mathbb{P}_n(\mathbf{x}), & \mathbf{x} = \mathbf{y}. \end{cases},$$

con

$$\mathbf{H}_n = \left\langle \mathbb{P}_n, \mathbb{P}_n^T \right\rangle = \left(\left\langle P^n_{\alpha(i)}, P^n_{\alpha(j)} \right\rangle\right)_{i,j=1}^{r_n^d}.$$

Polinomios Sobolev vs. Polinomios estándar

Un producto interno Sobolev (no estándar) $\langle \cdot, \cdot \rangle$ **ya no es mas simétrico con respecto al operador de multiplicación** debido a la presencia de derivadas. Por lo tanto,

- No hay una relación a tres términos
- No hay una fórmula de Christoffel-Darboux

  The deprival of this fundamental tool cannot be easily compensated. Different and ad hoc methods have been developed for dealing with different Sobolev inner products. The result is a theory of Sobolev orthogonal polynomials that appears fragmented and lack of uniformity. (Marcellán y Xu 2015)

“Multivariate Sobolev-type orthogonal polynomials”

Mello, Paschoa, Pérez y M. Piñar (2011) estudiaron los polinomios Sobolev con respecto a:

$$\langle f, g\rangle_S = \langle f, g\rangle_G + \lambda \nabla f(\mathbf{p}) \cdot \nabla g(\mathbf{p}), \quad \lambda > 0, \quad \mathbf{p} \in \mathbb{R}^d, \tag{2}$$

donde $\langle \cdot, \cdot\rangle_G$ es el producto interno:

$$\langle f, g\rangle_G = \int_G f(\mathbf{x}) g(\mathbf{x}) d\mu(\mathbf{x}). \tag{3}$$

$G \subset \mathbb{R}^d$ es un dominio con interior no vacío y $d\mu$ es una medida definida positiva sobre G.

“Multivariate Sobolev-type orthogonal polynomials”

- Sea $\{P_\alpha^n : |\alpha| = n\}$ una base ortonormal de $\mathscr{V}_n^d(G)$, el espacio de polinomios ortogonales con respecto a $\langle \cdot, \cdot \rangle_G$
- Sea $\mathbb{P}_n$ el vector:

$$\mathbb{P}_n(\mathbf{x}) := \left(P^n_{\alpha^{(1)}}(\mathbf{x}), P^n_{\alpha^{(2)}}(\mathbf{x}), \cdots, P^n_{\alpha^{(r_n^d)}}(\mathbf{x}) \right)^T,$$

donde $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(r_n^d)}$ es el ordenamiento de los elementos de $\left\{\alpha \in \mathbb{N}_0^d : |\alpha| = n\right\}$ usando orden lexicográfico inverso

- Sea $\nabla \mathbb{P}_n(\mathbf{x})$ la matriz de tamaño $r_n^d \times d$:

$$\nabla \mathbb{P}_n(\mathbf{x}) = \left(\partial_1 \mathbb{P}_n(\mathbf{x}), \partial_2 \mathbb{P}_n(\mathbf{x}), \cdots, \partial_d \mathbb{P}_n(\mathbf{x}) \right)$$

- Sea $K_n(\mathbf{x}, \mathbf{y})$ el kernel de Π_n^d:

$$K_n(\mathbf{x}, \mathbf{y}) = \sum_{j=0}^{n} \mathbb{P}_j^T(\mathbf{x}) (\mathbf{H}_j)^{-1} \mathbb{P}_j(\mathbf{y}), \quad \mathbf{H}_j = \left\langle \mathbb{P}_j, \mathbb{P}_j^T \right\rangle_G,$$

- Sean $\mathbf{K}_n^{(1,0)}$, $\mathbf{K}_n^{(0,1)}$ y $\mathbf{K}_n^{(1,1)}$, matrices de tamaños $d \times 1$, $1 \times d$, y $d \times d$, respectivamente:

$$\mathbf{K}_n^{(1,0)}(\mathbf{x},\mathbf{y}) = \sum_{j=0}^{n} (\nabla \mathbb{P}_j(\mathbf{x}))^T (\mathbf{H}_j)^{-1} \mathbb{P}_j(\mathbf{y}),$$

$$\mathbf{K}_n^{(0,1)}(\mathbf{x},\mathbf{y}) = \sum_{j=0}^{n} \mathbb{P}_j^T(\mathbf{x}) (\mathbf{H}_j)^{-1} \nabla \mathbb{P}_j(\mathbf{y}),$$

$$\mathbf{K}_n^{(1,1)}(\mathbf{x},\mathbf{y}) = \left(\partial_{x_i} \partial_{y_j} K_n(\mathbf{x},\mathbf{y})\right)_{i,j=1}^{d}.$$

“Multivariate Sobolev-type orthogonal polynomials”

Theorem

(Mello, Paschoa, Pérez y M. Piñar 2011, Theorem 3.1) Let $\{\mathbb{P}_n\}_{n\geq 0}$ *be an orthonormal polynomial system associated with* (3). *We define the polynomial system* $\{\mathbb{Q}_n\}_{n\geq 0}$ *by means of*

$$\mathbb{Q}_0(\mathbf{x}) := \mathbb{P}_0(\mathbf{x}),$$

$$\mathbb{Q}_n(\mathbf{x}) := \mathbb{P}_n(\mathbf{x}) - \lambda \nabla \mathbb{P}_n(\mathbf{p})[\mathbf{I}_d + \lambda \mathbf{K}_{n-1}^{(1,1)}(\mathbf{p},\mathbf{p})]^{-1}\mathbf{K}_{n-1}^{(1,0)}(\mathbf{p},\mathbf{x}), \quad n \geq 1. \tag{4}$$

Then $\{\mathbb{Q}_n\}_{n\geq 0}$ *is a sequence of orthogonal polynomials with respect to* $\langle\cdot,\cdot\rangle_S$ *defined in* (2). *Reciprocally, any sequence of orthogonal polynomials with respect to* (2) *can be expressed as in* (4).

Nota: Mello, Paschoa, Pérez y M. Piñar (2011, Lemma 2.1) mostraron que $\mathbf{I}_d + \lambda \mathbf{K}_n^{(1,1)}(\mathbf{p},\mathbf{p})$, $\lambda > 0$, $n \geq 0$, es una matriz $d \times d$ simétrica y no singular

“A higher order Sobolev-type inner product for orthogonal polynomials in several variables”

H. A. Dueñas, Garza y M. A. Piñar (2015) también estudiaron los polinomios ortogonales con respecto al producto interno Sobolev:

$$\langle f, g\rangle_S = \langle f, g\rangle_\sigma + M\nabla^{(j)} f(\xi)(\nabla^{(j)} g(\xi))^T, \quad \langle f, g\rangle_\sigma := \int_G f(\mathbf{x})g(\mathbf{x})d\sigma(\mathbf{x}). \tag{5}$$

- σ es una medida definida sobre un dominio $G \subseteq \mathbb{R}^d$ con interior no vacío,
- $\xi \in \mathbb{R}^d$,
- $\nabla^{(j)} f$ es el vector fila de las derivadas parciales de orden j de f,
- y $M \in \mathbb{R}_+$.

"A higher order Sobolev-type inner product for orthogonal polynomials in several variables"

Theorem

Let $\{\mathbb{P}_n\}_{n\geq 0}$ *be an orthonormal polynomial system associated with the inner product* $\langle\cdot,\cdot\rangle_\sigma$*. Define a Sobolev-type inner product as in* (5)*. Then, if we denote by* $\{\mathbb{Q}_n\}_{n\geq 0}$ *its corresponding OPS, normalized in such a way that* $\mathbb{Q}_n - \mathbb{P}_n$ *is a* r_n^d *dimensional vector whose components are polynomials of total degree lower than* n*, we have*

$$\mathbb{Q}_n(\mathbf{x}) = \begin{cases} \mathbb{P}_n(\mathbf{x}), & n < j, \\ \mathbb{P}_n(\mathbf{x}) - M\nabla^{(j)}\mathbb{P}_n(\xi)(I_{d^j} + MK_{n-1}^{(j,j)}(\xi,\xi))^{-1}K_{n-1}^{(j,0)}(\xi,\mathbf{x}), & n \geq j, \end{cases} \tag{6}$$

Conversely, if we define $\{\mathbb{Q}_n\}_{n\geq 0}$ *as in* (6)*, then they are an OPS with respect to* (5)*.*

- $\nabla^{(j)}\mathbb{P}_n$ es la matriz de tamaño $r_n^d \times d^j$ de las derivadas de orden j de $\mathbb{P}_n$,
- $K_n(\mathbf{x},\mathbf{y})$ es el polinomio kernel de de Π_n^d,
- $K_n^{(j,j)}(\mathbf{x},\mathbf{y})$ es la matriz de tamaño $d^j \times d^j$ con todas las derivadas de orden j de $K_n(\mathbf{x},\mathbf{y})$.

“A family of Sobolev orthogonal polynomials on the unit ball”

Xu (2006) consideró los polinomios ortogonales Sobolev sobre la bola $\mathbb{B}^d = \left\{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| \leq 1\right\}$ con respecto a:

$$\langle f, g\rangle_{\triangle} = \frac{1}{4d^2 \operatorname{vol}(\mathbb{B}^d)} \int_{\mathbb{B}^d} \triangle \left[(1 - \|\mathbf{x}\|^2) f(\mathbf{x})\right] \triangle \left[(1 - \|\mathbf{x}\|^2) g(\mathbf{x})\right] d\mathbf{x}, \tag{7}$$

donde $\triangle$ es el Laplaciano, $\operatorname{vol}(\mathbb{B}^d)$ es el volumen de $\mathbb{B}^d$:

$$\operatorname{vol}(\mathbb{B}^d) = \frac{2\pi^{d/2}}{d\Gamma(d/2)}.$$

Motivación: El trabajo de Xu (2006) fue motivado por el artículo de Atkinson y Hansen (2005) (caso $d = 2$ y la solución numérica de la ecuación de Poisson $-\triangle u = f(\cdot, u)$).

“A family of Sobolev orthogonal polynomials on the unit ball”

Theorem

(Xu 2006, Theorem 2.4) A mutually orthogonal basis for $\mathscr{V}_n^d(\triangle)$ is given by:

$$Q_{0,\nu}^n(\mathbf{x}) = Y_\nu^n(\mathbf{x}),$$

$$Q_{j,\nu}^n(\mathbf{x}) = (1-\|\mathbf{x}\|^2)P_{j-1}^{(2,n-2j+\frac{d-2}{2})}(2\|\mathbf{x}\|^2-1)Y_\nu^{n-2j}(\mathbf{x}), \quad 1 \le j \le n/2,$$

where $\left\{Y_\nu^{n-2j} : 1 \le \nu \le a_{n-2j}^d\right\}$ *is an orthonormal basis of* $\mathscr{H}_{n-2j}^d$. *Furthermore,*

$$\left\langle Q_{0,\nu}^n, Q_{0,\nu}^n \right\rangle_\triangle = \frac{2n+d}{d}, \quad \left\langle Q_{j,\nu}^n, Q_{j,\nu}^n \right\rangle_\triangle = \frac{8j^2(j+1)^2}{d(n+d/2)}.$$

Nota: $P_n^{(\alpha,\beta)}(t)$, $\alpha, \beta > -1$, es el polinomio de Jacobi de grado n.

“Sobolev orthogonal polynomials defined via gradient on the unit ball”

Xu (2008) estudió los polinomios Sobolev sobre la bola con respecto a:

$$\langle f, g \rangle_I = \frac{\lambda}{\omega_{d-1}} \int_{\mathbb{B}^d} \nabla f(\mathbf{x}) \cdot \nabla g(\mathbf{x}) d\mathbf{x} + \frac{1}{\omega_{d-1}} \int_{\mathbb{S}^{d-1}} f(\xi) g(\xi) d\omega(\xi), \quad (8)$$

$$\langle f, g \rangle_{\|} = \frac{\lambda}{\omega_{d-1}} \int_{\mathbb{B}^d} \nabla f(\mathbf{x}) \cdot \nabla g(\mathbf{x}) d\mathbf{x} + f(\mathbf{0}) g(\mathbf{0}), \quad (9)$$

donde $\lambda > 0$ y ω_{d-1} es el área de la esfera $\mathbb{S}^{d-1} = \left\{ \mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| = 1 \right\}$

$$\omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}.$$

"Sobolev orthogonal polynomials defined via gradient on the unit ball"

Theorem

(Xu 2008, Theorem 2.3) A mutually orthogonal basis

$$\left\{U_{j,\nu}^{n} : 0 \leq j \leq n/2, 1 \leq \nu \leq a_{n-2j}^{d}\right\}$$

for $\mathscr{V}_n^d(I)$ *is given by:*

$$U_{0,\nu}^{n}(\mathbf{x}) = Y_{\nu}^{n}(\mathbf{x}),$$

$$U_{j,\nu}^{n}(\mathbf{x}) = (1-\|\mathbf{x}\|^2) P_{j-1}^{(1,n-2j+\frac{d-2}{2})}(2\|\mathbf{x}\|^2 - 1) Y_{\nu}^{n-2j}(\mathbf{x}), \quad 1 \leq j \leq n/2,$$

where $\left\{Y_{\nu}^{n-2j} : 1 \leq \nu \leq a_{n-2j}^{d}\right\}$ *is an orthonormal basis of* $\mathscr{H}_{n-2j}^{d}$*. Furthermore,*

$$\left\langle U_{0,\nu}^{n}, U_{0,\nu}^{n}\right\rangle_I = n\lambda + 1, \quad \left\langle U_{j,\nu}^{n}, U_{j,\nu}^{n}\right\rangle_I = \frac{2j^2}{n+\frac{d-2}{2}}\lambda.$$

“Sobolev orthogonal polynomials defined via gradient on the unit ball”

Theorem

(Xu 2008, Theorem 2.6) A mutually orthogonal basis

$$\left\{V_{j,\nu}^n : 0 \le j \le n/2, 1 \le \nu \le a_{n-2j}^d\right\}$$

for $\mathscr{V}_n^d(\|)$ *is given by:*

$$V_{j,\nu}^n(\mathbf{x}) = U_{j,\nu}^n(\mathbf{x}), \quad 0 \le j \le \left\lfloor \frac{n-1}{2} \right\rfloor,$$

$$V_{n/2}^n(\mathbf{x}) = \frac{4}{n+d-2}\left(P_{n/2}^{(-1,\frac{d-2}{2})}(2\|\mathbf{x}\|^2-1) - (-1)^{n/2}\frac{(d/2)_{n/2}}{(n/2)!}\right),$$

where $V_{n/2}^n(\mathbf{x}) := V_{n/2,\nu}^n(\mathbf{x})$ *holds only when* n *is even. Furthermore,*

$$\left\langle V_{j,\nu}^n, V_{j,\nu}^n\right\rangle_{\|} = \left\langle U_{j,\nu}^n, U_{j,\nu}^n\right\rangle_I, \quad 0 \le j \le \left\lfloor \frac{n-1}{2} \right\rfloor, \quad \left\langle V_{n/2}^n, V_{n/2}^n\right\rangle_{\|} = \frac{8\lambda}{n+\frac{d-2}{2}}.$$

“Orthogonal polynomials and partial differential equations on the unit ball”

Se sabe que los polinomios en $\mathscr{V}_n^d(W_\mu)$, $W_\mu(\mathbf{x}) := (1 - \|\mathbf{x}\|^2)^\mu$, $\mu > -1$, sobre la bola $\mathbb{B}^d$, son ortogonales con respecto a:

$$\langle f, g\rangle_\mu = c_\mu \int_{\mathbb{B}^d} f(\mathbf{x}) g(\mathbf{x}) W_\mu(\mathbf{x}) d\mathbf{x}, \tag{10}$$

$$c_\mu := \left(\int_{\mathbb{B}^d} W_\mu(\mathbf{x}) d\mathbf{x} \right)^{-1} = \frac{\Gamma(\mu + d/2 + 1)}{\pi^{d/2}\Gamma(\mu + 1)}, \tag{11}$$

Una base mutuamente ortogonal $\left\{ P_{j,\nu}^n : 0 \le j \le n/2, 1 \le \nu \le a_{n-2j}^d \right\}$ de $\mathscr{V}_n^d(W_\mu)$ es

$$P_{j,\nu}^n(W_\mu; \mathbf{x}) = P_j^{(\mu, n-2j+\frac{d-2}{2})}(2\|\mathbf{x}\|^2 - 1) Y_\nu^{n-2j}(\mathbf{x}), \tag{12}$$

donde $\left\{ Y_\nu^{n-2j} : 1 \le \nu \le a_{n-2j}^d \right\}$ denota una base ortonormal de $\mathscr{H}_{n-2j}^d$.

"Orthogonal polynomials and partial differential equations on the unit ball"

Se sabe que si $P \in \mathscr{V}_n^d(W_\mu)$ entonces

$$\mathcal{L}_\mu P = -n(n+2\mu+d)P, \quad \mu > -1, \quad P \in \mathscr{V}_n^d(W_\mu), \tag{13}$$

donde

$$\mathcal{L}_\mu := \triangle - \left\langle \mathbf{x}^T, \nabla \right\rangle^2 - (2\mu+d)\left\langle \mathbf{x}^T, \nabla \right\rangle, \quad \left\langle \mathbf{x}^T, \nabla \right\rangle = \sum_{i=1}^{d} x_i \partial_i. \tag{14}$$

“Orthogonal polynomials and partial differential equations on the unit ball”

M. A. Piñar y Xu (2009) estudiaron la ecuación diferencial

$$\mathcal{L}_\mu P = -n(n + 2\mu + d)P, \tag{15}$$

para $\mu = -1, -2, -3, -4, \ldots$.

Theorem

(M. A. Piñar y Xu 2009, Theorem 3.3) Elements of $\mathscr{V}_n^d(I)$ *satisfy* $\mathcal{L}_{-1}P = -n(n + d - 2)P$. *In particular, the eigenfunctions of the operator* $\mathcal{L}_{-1}$ *consist of a complete polynomial basis.*

“Orthogonal polynomials and partial differential equations on the unit ball”

Para $\mu = -k$, $k = 2, 3, \ldots$ M. A. Piñar y Xu definieron:

$$\mathscr{U}_n^d(W_{-k}) := \mathscr{H}_n^d \cup \left(\bigcup_{j=1}^{k-1} \left[\sum_{\nu=0}^{j} a_{j,\nu}^n (1 - \|\mathbf{x}\|^2)^\nu \right] \mathscr{H}_{n-2j}^d \right) \cup (1 - \|\mathbf{x}\|^2)^k \mathscr{V}_{n-2k}^d(W_k),$$

donde, para $1 \leq j \leq k-1$,

$$a_{j,\nu}^n := \frac{(-1)^{j-\nu} j! (-k+1)_j (n-j-k+d/2)_\nu}{\nu! (j-\nu)! (-k+1)_\nu (n-j-k+d/2)_j}, \quad 0 \leq \nu \leq j,$$

y donde $a_{j,\nu}^n$ está bien definido si $n - j - k + \nu + d/2 \neq 0$.

“Orthogonal polynomials and partial differential equations on the unit ball”

Theorem

(M. A. Piñar y Xu 2009, Theorem 3.4) If $\mu = -k$ *and* $k = 2, 3, \ldots$, *then the polynomials in* $\mathscr{U}_n^d(W_{-k})$ *satisfy equation* (15)*; that is,* $\mathcal{L}_{-k}P = -n(n-2k+d)P$ *for* $P \in \mathscr{U}_n^d(W_{-k})$*. Furthermore,*

$$\dim \mathscr{U}_n^d = \dim \mathscr{P}_n^d, \quad \textit{if } n - 2k - 1 + d/2 \neq 0.$$

In particular, the operator $\mathcal{L}_{-k}$ *has a complete polynomial basis of eigenfunctions if the dimension* d *is odd.*

“Orthogonal polynomials and partial differential equations on the unit ball”

M. A. Piñar y Xu (2009, Theorem 4.1) observaron que para $\mu = -2$, los polinomios en

$$\mathscr{V}_n^d(W_{-2}) := \mathscr{H}_n^d \cup (1 - \|\mathbf{x}\|^2)\mathscr{H}_{n-2}^d \cup (1 - \|\mathbf{x}\|^2)^2 \mathscr{V}_{n-4}^d(W_2),$$

son ortogonales con respecto a:

$$\langle f, g\rangle_{-2} = \frac{\lambda}{\omega_{d-1}} \int_{\mathbb{B}^d} \triangle f(\mathbf{x}) \triangle g(\mathbf{x}) d\mathbf{x} + \frac{1}{\omega_{d-1}} \int_{\mathbb{S}^{d-1}} f(\mathbf{x}) g(\mathbf{x}) d\omega(\mathbf{x}), \quad \lambda > 0.$$

Theorem

(M. A. Piñar y Xu 2009, Theorem 4.1) The elements in $\mathscr{V}_n^d(W_{-2})$ are orthogonal polynomials with respect to $\langle \cdot, \cdot \rangle_{-2}$. Moreover, they contain an orthonormal basis; in other words,

$$\mathscr{V}_n^d(W_{-2}) = \mathscr{H}_n^d \oplus (1 - \|\mathbf{x}\|^2)\mathscr{H}_{n-2}^d \oplus (1 - \|\mathbf{x}\|^2)^2 \mathscr{V}_{n-4}^d(W_2).$$

“Orthogonal polynomials and partial differential equations on the unit ball”

Con respecto al producto interno:

$$\langle f,g\rangle_n^* = \frac{\lambda_1}{\omega_{d-1}}\int_{\mathbb{B}^d} \triangle f(\mathbf{x}) \triangle g(\mathbf{x}) d\mathbf{x} + \frac{1}{\omega_{d-1}}\int_{\mathbb{S}^{d-1}} f(\mathbf{x})g(\mathbf{x})d\omega(\mathbf{x})$$
$$+ \frac{\lambda_2}{\omega_{d-1}}\int_{\mathbb{S}^{d-1}} \frac{d}{dr}[r^{n-4-d}f(\mathbf{x})]\frac{d}{dr}[r^{n-4-d}g(\mathbf{x})]d\omega(\mathbf{x}), \quad \lambda_1,\lambda_2>0,$$

donde d/dr es la derivada normal otro resultado es el siguiente:

Theorem

(M. A. Piñar y Xu 2009, Theorem 4.2) The elements of $\mathscr{V}_n^d(W_{-2})$ *are orthogonal polynomials with respect to the inner product* $\langle f,g\rangle_n^*$.

Li y Xu (2014) estudiaron los polinomios en $\mathscr{V}_n^d(W_{-s})$, $s \in \mathbb{N}$, el espacio de polinomios con respecto a:

$$\langle f, g \rangle_{-s} = \langle \nabla^s f, \nabla^s g \rangle_{\mathbb{B}^d} + \sum_{i=0}^{\lceil s/2 \rceil - 1} \lambda_i \left\langle \triangle^i f, \triangle^i g \right\rangle_{\mathbb{S}^{d-1}}, \quad s = 1, 2, 3 \ldots, \tag{16}$$

con $\lambda_i > 0$ para $i = 0, 1, \ldots, \lceil s/2 \rceil - 1$,

$$\nabla^{2m} := \triangle^m, \quad \text{y} \quad \nabla^{2m+1} := \nabla \triangle^m, \quad m = 1, 2, 3, \ldots,$$

y donde $\langle \cdot, \cdot \rangle_{\mathbb{B}^d} := \langle \cdot, \cdot \rangle_{\mu=0}$.

Li y Xu (2014, Proposition 2.1, Definition A.2) extendieron los polinomios $P^{n}_{j,\nu}(W_\mu; \mathbf{x})$, ortogonales con respecto a W_μ, a:

$$P^{\mu,n}_{j,l}(\mathbf{x}) := (n-j+d/2)_j J_j^{\mu, n-2j+\frac{d-2}{2}}(2\|\mathbf{x}\|^2 - 1) Y_l^{n-2j}(\mathbf{x}),$$
$$\mu \in \mathbb{R}, \quad n \in \mathbb{N}_0, \quad 0 \le j \le n/2, \quad 1 \le l \le a^d_{n-2j},$$

donde $J_j^{\alpha,\beta}(t)$, $\alpha, \beta \in \mathbb{R}$, es el polinomios generalizado de Jacobi de grado j y $\left\{ Y_l^{n-2j} : 1 \le l \le a^d_{n-2j} \right\}$ es una base ortonormal de $\mathscr{H}^d_{n-2j}$.

Para $s \in \mathbb{N}$, $n \in \mathbb{N}_0$, $0 \leq j \leq n/2$, $1 \leq l \leq a^d_{n-2j}$, $\mathbf{x} \in \mathbb{B}^d$, y $\xi \in \mathbb{S}^{d-1}$, se define:

$$Q^{-s,n}_{j,l}(\mathbf{x}) = \begin{cases} P^{-s,n}_{j,l}(\mathbf{x}), & j \geq s, \\ P^{-s,n}_{j,l}(\mathbf{x}) - \displaystyle\sum_{k=0}^{\lceil s/2 \rceil - 1} \frac{\triangle^k P^{-s,n}_{j,l}(\xi)}{Y^{n-2j}_l(\xi)} Y^{n-2j,k}_l(\mathbf{x}), & \lceil s/2 \rceil \leq j \leq s-1, \\ Y^{n-2j,j}_l(\mathbf{x}), & 0 \leq j \leq \lceil s/2 \rceil - 1, \end{cases}$$

donde para $n, j \in \mathbb{N}_0$, $Y^{n,j}_l(\mathbf{x}) := 0$ si $j < 0$ y, si $j \geq 0$,

$$Y^{n,j}_l(\mathbf{x}) := \sum_{i=0}^{j} c^{n,j}_i (1 - \|\mathbf{x}\|^2)^i Y^n_l(\mathbf{x}), \quad 1 \leq l \leq a^d_n,$$

y $c^{n,j}_i$, $0 \leq i \leq j$, es la única solución del sistema de ecuaciones:

$$4^k \sum_{i=k}^{j} (-i)_k (-k)_{i-k} \frac{(n + d/2)_k}{(n + d/2)_{i-k}} c^{n,j}_i = \delta_{k,j}, \quad 0 \leq k \leq j.$$

Theorem

(Li y Xu 2014, Theorem 3.7) The polynomials in

$$\left\{Q_{j,l}^{-s,n}(\mathbf{x}) : 0 \leq j \leq n/2, 1 \leq l \leq a_{n-2j}^{d}\right\}$$

form an orthogonal basis of $\mathscr{V}_n^d(W_{-s})$. *More precisely,* $\left\langle Q_{j,l}^{-s,n}, Q_{i,k}^{-s,m}\right\rangle_{-s} = h_{j,l}^{-s}\delta_{n,m}\delta_{j,i}\delta_{l,k}$ *for* $\langle\cdot,\cdot\rangle_{-s}$ *defined in* (16), *where:*

$$h_{j,l}^{-s} = \begin{cases} 2^{2s-1}d(n+d/2-s)_s(n+d/2-s+1)_{s-1}, & j \geq \lceil s/2 \rceil, \\ d(n-2j)+\lambda_j, & j=(s-1)/2, \\ \lambda_j, & 0 \leq j \leq (s-1)/2. \end{cases}$$

“Weighted Sobolev orthogonal polynomials on the unit ball”

Pérez, M. A. Piñar y Xu (2013) estudiaron el producto interno Sobolev,

$$\langle f, g\rangle_{\nabla, W_\mu} := \frac{\lambda}{\omega_{d-1}} \int_{\mathbb{B}^d} \nabla f(\mathbf{x}) \cdot \nabla g(\mathbf{x}) W_{\mu+1}(\mathbf{x}) d\mathbf{x} + \frac{1}{\omega_{d-1}} \int_{\mathbb{S}^{d-1}} f(\xi) g(\xi) d\omega(\xi), \quad \mu > -1, \quad \lambda > 0, \quad (17)$$

“Weighted Sobolev orthogonal polynomials on the unit ball”

Theorem

(Pérez, M. A. Piñar y Xu 2013, Theorem 3.4) For $0 \leq j \leq n/2$, *let* $\left\{Y_\nu^{n-2j} : 1 \leq \nu \leq a_{n-2j}^d\right\}$ *be an orthonormal basis of* $\mathscr{H}_{n-2j}^d$. *Define*

$$R_{0,\nu}^n(\mathbf{x}) = Y_\nu^n(\mathbf{x}),$$

$$R_{j,\nu}^n(\mathbf{x}) = \left[P_j^{(\mu,n-2j+\frac{d-2}{2})}(2\|\mathbf{x}\|^2 - 1) - P_j^{(\mu,n-2j+\frac{d-2}{2})}(1)\right] Y_\nu^{n-2j}(\mathbf{x}), \quad 1 \leq j \leq n$$

Then $\left\{R_{j,\nu}^n : 0 \leq j \leq n/2, 1 \leq \nu \leq a_{n-2j}^d\right\}$ *forms a mutually orthogonal basis of* $\mathscr{V}_n^d(\nabla, W_\mu)$. *Furthermore,*

$$\left\langle R_{0,\nu}^n, R_{0,\nu}^n \right\rangle_{\nabla,W_\mu} = \lambda n \frac{\Gamma(\mu+2)\Gamma(n+d/2)}{\Gamma(n+\mu+1+d/2)} + 1,$$

$$\left\langle R_{j,\nu}^n, R_{j,\nu}^n \right\rangle_{\nabla,W_\mu} = \lambda(n(2j+\mu+1) - j(2j-d+2)) \times \frac{\Gamma(\mu+j+1)\Gamma(n-j+d/2)(n+\mu-j+d/2)}{j!\Gamma(n+\mu+1-j+d/2)(n+\mu+d/2)}$$

“Weighted Sobolev orthogonal polynomials on the unit ball”

Theorem

(Pérez, M. A. Piñar y Xu 2013, Theorem 3.5) Let $\mu > -1$ *and let* $P^n_{j,\nu}$ *be the mutually orthogonal polynomials in* $\mathscr{V}^d_n(W_\mu)$*, defined in* (12)*. Then they are also mutually orthogonal with respect to the Sobolev inner product:*

$$\langle f, g\rangle = c_\mu \left[\int_{\mathbb{B}^d} f(\mathbf{x})g(\mathbf{x})W_\mu(\mathbf{x})d\mathbf{x} + \lambda \int_{\mathbb{B}^d} \nabla f(\mathbf{x})\cdot\nabla g(\mathbf{x})W_{\mu+1}(\mathbf{x})d\mathbf{x}\right], \tag{18}$$

where $\lambda > 0$ *is a fixed constant.*

Comentario: Notar el cambio en los parámetros de los dos pesos: W_μ y $W_{\mu+1}$

“Weighted Sobolev orthogonal polynomials on the unit ball”

El resultado principal en (Pérez, M. A. Piñar y Xu 2013, Theorem 5.2) es con respecto al producto interno:

$$\langle f, g\rangle_{\nabla, W_\mu, \mathbb{B}^d} = c_\mu \left[\int_{\mathbb{B}^d} f(\mathbf{x}) g(\mathbf{x}) W_\mu(\mathbf{x}) d\mathbf{x} + \lambda \int_{\mathbb{B}^d} \nabla f(\mathbf{x}) \cdot \nabla g(\mathbf{x}) W_\mu(\mathbf{x}) d\mathbf{x}\right], \quad (19)$$

con $\lambda > 0$ y $\mu > -1$.

"Weighted Sobolev orthogonal polynomials on the unit ball"

Theorem

(Pérez, M. A. Piñar y Xu 2013, Theorem 5.2) Let $\lambda > 0$*. For* $0 \leq j \leq n/2$*, let* $\beta_j := n - 2j + \frac{d-2}{2}$ *and let* $q_k^{(\mu,\beta_j)}(t)$ *be the* k*-th Sobolev orthogonal polynomial associated with the inner product* $\langle \cdot, \cdot \rangle_{\mu,\beta_j}$ *in (20). Let* $\left\{ Y_\nu^{n-2j} : 1 \leq \nu \leq a_{n-2j}^d \right\}$ *be an orthonormal basis of* $\mathscr{H}_{n-2j}^d$*. Define*

$$T_{j,\nu}^n(\mathbf{x}) := q_j^{(\mu,\beta_j)}(2\|\mathbf{x}\|^2 - 1) Y_\nu^{n-2j}(\mathbf{x}).$$

Then the set $\left\{ T_{j,\nu}^n : 0 \leq j \leq n/2, 1 \leq \nu \leq a_{n-2j}^d \right\}$ *is a mutually orthogonal basis of* $\mathscr{V}_n^d(\nabla, W_\mu, \mathbb{B}^d)$*. Moreover,*

$$\left\langle T_{j,\nu}^n, T_{j,\nu}^n \right\rangle_{\nabla, W_\mu, \mathbb{B}^d} := \frac{\Gamma(\mu + 1 + d/2)}{\Gamma(\mu+1)\Gamma(d/2)2^{\beta_j + \mu}} \left\langle q_j^{(\mu,\beta_j)}, q_j^{(\mu,\beta_j)} \right\rangle_{\mu,\beta_j}.$$

“Weighted Sobolev orthogonal polynomials on the unit ball”

Donde $\langle f, g\rangle_{\alpha,\beta}$ es:

$$\langle f, g\rangle_{\alpha,\beta} := \int_{-1}^{1} f(x)g(x)w_{\alpha,\beta}(x)dx+$$
$$2\lambda \int_{-1}^{1} (f, f') \begin{pmatrix} \beta(2\beta - d + 2) & (2\beta - d + 2)(1 + x) \\ (2\beta - d + 2)(1 + x) & 4(1 + x)^2 \end{pmatrix} \begin{pmatrix} g \\ g' \end{pmatrix} w_{\alpha,\beta-1}(x)dx, \tag{20}$$

donde $d \in \mathbb{N}$, $\lambda > 0$, $\alpha > -1$, $\beta > \max\{0, (d - 2)/2\}$, y $w_{\alpha,\beta}(x) = (1 - x)^{\alpha}(1 + x)^{\beta}$ es el peso Jacobi.

Recientemente Fernández, Marcellán, Pérez, M. Piñar y Xu (2015) estudiaron los polinomios en dos variables con respecto a:

$$\langle f,g\rangle_S = c\int_\Omega \nabla f(x,y)\cdot\nabla g(x,y)W(x,y)dxdy + \lambda f(c_1,c_2)g(c_1,c_2),$$

$$(c_1,c_2)\in\mathbb{R}^2, \quad c = \left(\int_\Omega W(x,y)dxdy\right)^{-1}, \quad (21)$$

con $W(x,y) = w_1(x)w_2(y)$ y $\Omega := [a_1,b_1]\times[a_2,b_2]$ es un dominio producto.
Si $\langle\cdot,\cdot\rangle_\nabla$ denota la forma bilineal:

$$\langle f,g\rangle_\nabla = c\int_\Omega \nabla f(x,y)\cdot\nabla g(x,y)W(x,y)dxdy. \quad (22)$$

entonces:

Proposition

(Fernández, Marcellán, Pérez, M. Piñar y Xu 2015, Proposition 2.3) For $n \geq 1$*, let* $\{S_k^n : 0 \leq k \leq n\}$ *denote a monic orthogonal basis of* $\mathscr{V}_n^2(\nabla)$*. Then, the monic orthogonal basis* $\{\mathcal{S}_k^n : 0 \leq k \leq n\}$ *of* $\mathscr{V}_n^2(S)$ *is given by* $\mathcal{S}_0^0(x,y) = 1$ *and*

$$\mathcal{S}_k^n(x,y) = S_k^n(x,y) - S_k^n(c_1,c_2), \quad n \geq 1.$$

Para construir una base para $\mathscr{V}_n^2(\nabla)$, definieron

$$Q_k^n(x,y) = q_{n-k}(w_1;x)q_k(w_2;y), \quad 0 \leq k \leq n, \quad n = 0,1,2,\ldots, \quad (23)$$

donde $q_n(w_i)$, $i = 1,2$, es mónico y dado por:

$$q_n(w_i;x) = p_n(w_i;x) + na_{n-1}(w_i)p_{n-1}(w_i;x) + nb_{n-1}(w_i)p_{n-2}(w_i;x), \quad n \geq$$

el cual satisface $q_n'(w_i) = np_{n-1}(w_i)$, con $\{p_n(w_i;x)\}_{n\geq 0}$ una sucesión auto-coherente:

$$p_n(w_i;x) = \frac{p_{n+1}'(w_i;x)}{n+1} + a_n(w_i)p_n'(w_i;x) + b_n(w_i)p_{n-1}'(w_i;x), \quad n \geq 1, \quad (24)$$

Se define luego

$$\mathbb{Q}_n = \left(Q_0^n(x,y), Q_1^n(x,y), \ldots, Q_n^n(x,y)\right)^T,$$
$$\mathbb{S}_n = \left(S_0^n(x,y), S_1^n(x,y), \ldots, S_n^n(x,y)\right)^T$$

En $\mathbb{S}_n$ están los polinomios en la base $\{S_k^n : 0 \leq k \leq n\}$ de $\mathscr{V}_n^2(\nabla)$.

Proposition

(Fernández, Marcellán, Pérez, M. Piñar y Xu 2015, theorem 2.5) There exist matrices $\mathbf{A}_n$ and $\mathbf{B}_n$ such that:

$$\mathbb{Q}_n \overset{c}{=} \mathbb{S}_n + \mathbf{A}_{n-1}\mathbb{S}_{n-1} + \mathbf{B}_{n-2}\mathbb{S}_{n-2}, \tag{25}$$

El símbolo $\overset{c}{=}$ denota una relación de congruencia sobre Π^2:
$P(x,y) \overset{c}{=} Q(x,y)$ if $P(x,y) - Q(x,y) = c$.

“Sobolev orthogonal polynomials of several variables on product domains”

Recientemente H. Dueñas, Salazar-Morales y M. Piñar (2021) estudiaron los polinomios ortogonales con respecto:

$$\langle f,g\rangle_S = c\int_\Omega \nabla^\kappa f(\mathbf{x})\cdot\nabla^\kappa g(\mathbf{x})W(\mathbf{x})d\mathbf{x} + \sum_{i=0}^{\kappa-1}\lambda_i\nabla^i f(\mathbf{p})\cdot\nabla^i g(\mathbf{p}), \quad (26)$$

con $\mathbf{p}\in\mathbb{R}^d$, $\lambda_i>0$ for $i=0,1,\ldots,\kappa-1$, $W(\mathbf{x})=w_1(x_1)\cdots w_d(x_d)$ una función de pesono negativa sobre $\Omega=[a_1,b_1]\times\cdots\times[a_d,b_d]$, c es la constante de normalización de W:

$$c := \left(\int_\Omega W(\mathbf{x})d\mathbf{x}\right)^{-1},$$

“Sobolev orthogonal polynomials of several variables on product domains”

y $\nabla^\kappa f$ es el vector columna:

$$\nabla^\kappa f := \begin{pmatrix} \partial_1(\nabla^{\kappa-1} f) \\ \partial_2(\nabla^{\kappa-1} f) \\ \vdots \\ \partial_d(\nabla^{\kappa-1} f) \end{pmatrix}, \quad \kappa \geq 1, \quad \text{where} \quad \nabla^0 f := f. \tag{27}$$

Si $\langle \cdot, \cdot \rangle_{\nabla^\kappa}$ denota la forma bilineal:

$$\langle f, g \rangle_{\nabla^\kappa} := c \int_\Omega \nabla^\kappa f(\mathbf{x}) \cdot \nabla^\kappa g(\mathbf{x}) W(\mathbf{x}) d\mathbf{x}, \tag{28}$$

entonces:

“Sobolev orthogonal polynomials of several variables on product domains”

Theorem

Let $\{S^n_\alpha : |\alpha| = n\}$ *denote a monic orthogonal basis of* $\mathscr{V}^d_n(\nabla^\kappa)$*. Then, a monic orthogonal basis* $\{\mathcal{S}^n_\alpha : |\alpha| = n\}$ *of* $\mathscr{V}^d_n(S)$ *is given by:*

$$\mathcal{S}^n_\alpha(\mathbf{x}) = (\mathbf{x} - \mathbf{p})^\alpha, \quad 0 \leq |\alpha| = n < \kappa, \tag{29}$$

$$\mathcal{S}^n_\alpha(\mathbf{x}) = S^n_\alpha(\mathbf{x}) - \mathcal{T}^{\kappa-1}(S^n_\alpha, \mathbf{p}; \mathbf{x}), \quad |\alpha| = n \geq \kappa, \tag{30}$$

where $(\mathbf{x} - \mathbf{p})^\alpha$ *denotes the shifted monomial* $(x_1 - p_1)^{\alpha_1}(x_2 - p_2)^{\alpha_2} \cdots (x_d - p_d)^{\alpha_d}$ *and* $\mathcal{T}^{\kappa-1}(S^n_\alpha, \mathbf{p}; \mathbf{x})$ *denotes the Taylor polynomial of total degree* $\kappa - 1$ *in* d *variables of* S^n_α *at* $\mathbf{p} = (p_1, p_2, \ldots, p_d)$*.*

Recordar que:

$$\mathcal{T}^{\kappa-1}(P, \mathbf{p}; \mathbf{x}) = \sum_{|\beta| \leq \kappa - 1} \frac{(\partial^\beta P)(\mathbf{p})}{\beta!}(\mathbf{x} - \mathbf{p})^\beta$$

“Sobolev orthogonal polynomials of several variables on product domains”

Si $\mathbb{S}_n$ denota al vector: $\mathbb{S}_n(\mathbf{x}) = \left(S^n_{\alpha^{(1)}}(\mathbf{x}), S^n_{\alpha^{(2)}}(\mathbf{x}), \ldots, S^n_{\alpha^{(r^d_n)}}(\mathbf{x})\right)^T$ de los elementos en la base $\{S^n_\alpha : |\alpha| = n\}$ de $\mathscr{V}^d_n(\nabla^\kappa)$, arreglados usando orden lexicográfico inverso, y si $\mathbb{Q}_n$ denota un vector de polinomios mónicos:

$$\mathbb{Q}_n = \left(Q^n_{\alpha^{(1)}}(\mathbf{x}), Q^n_{\alpha^{(2)}}(\mathbf{x}), \ldots, Q^n_{\alpha^{(r^d_n)}}(\mathbf{x})\right)^T,$$

elegido apropiadamente:

Proposition

There exist real matrices $\mathbf{A}_{n,i}$ of size $r^d_n \times r^d_i$, $\kappa \le i \le n-1$, such that:

$$\mathbb{Q}_n \stackrel{\kappa-1}{=} \mathbb{S}_n, \quad n \le \kappa, \quad \textit{and} \quad \mathbb{Q}_n \stackrel{\kappa-1}{=} \mathbb{S}_n + \sum_{i=\kappa}^{n-1} \mathbf{A}_{n,i}\mathbb{S}_i, \quad n > \kappa.$$

El símbolo $\stackrel{\kappa-1}{=}$ denota una relación de congruencia sobre Π^d: $P(\mathbf{x}) \stackrel{\kappa-1}{=} Q(\mathbf{x})$ si $P(\mathbf{x}) - Q(\mathbf{x}) \in \Pi^d_{\kappa-1}$.

Referencias I

Atkinson, Kendall y Olaf Hansen (2005). "Solving the nonlinear Poisson equation on the unit disk". En: *J. Integral Equations Appl.* 17.3, págs. 223-241.

Dueñas, Herbert A., Luis E. Garza y Miguel A. Piñar (2015). "A higher order Sobolev-type inner product for orthogonal polynomials in several variables". En: *Numer. Algorithms* 68.1, págs. 35-46.

Dueñas, Herbert A., Omar Salazar-Morales y Miguel A. Piñar (2021). "Sobolev orthogonal polynomials of several variables on product domains". En: *Mediterr. J. Math.* 18.5. Article 227, págs. 1-21.

Fernández, Lidia, Francisco Marcellán, Teresa E. Pérez, Miguel A. Piñar y Yuan Xu (2015). "Sobolev orthogonal polynomials on product domains". En: *J. Comput. Appl. Math.* 284, págs. 202-215.

Li, Huiyuan y Yuan Xu (2014). "Spectral approximation on the unit ball". En: *SIAM J. Numer. Anal.* 52.6, págs. 2647-2675.

Marcellán, Francisco y Yuan Xu (2015). "On Sobolev orthogonal polynomials". En: *Expo. Math.* 33, págs. 308-352.

Referencias II

Mello, Mirela V., Vanessa G. Paschoa, Teresa E. Pérez y Miguel A. Piñar (2011). "Multivariate Sobolev-type orthogonal polynomials". En: *Jaen Journal on Approximation* 3.2, págs. 241-259.

Pérez, Teresa E., Miguel A. Piñar y Yuan Xu (2013). "Weighted Sobolev orthogonal polynomials on the unit ball". En: *J. Approx. Theory* 171, págs. 84-104.

Piñar, Miguel A. y Yuan Xu (sep. de 2009). "Orthogonal polynomials and partial differential equations on the unit ball". En: *Proc. Amer. Math. Soc.* 137.9, págs. 2979-2987.

Xu, Yuan (2006). "A family of Sobolev orthogonal polynomials on the unit ball". En: *J. Approx. Theory* 138, págs. 232-241.

– (2008). "Sobolev orthogonal polynomials defined via gradient on the unit ball". En: *J. Approx. Theory* 152, págs. 52-65.

Preguntas