Seminario Iberoamericano de Análisis Matemático y Matemática Aplicada

Sobolev Spaces with Measures

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• In the context of Sobolev orthogonal polynomials there is no such a thing as the usual three term recurrence relation for orthogonal polynomials in L^2. Therefore, usually it is not possible to find an explicit expression for the Sobolev orthogonal polynomial of degree n.

• Hence, it is especially important to count with an asymptotic estimate for the behavior of Sobolev orthogonal polynomials.

• In fact, we also study extremal polynomials for every exponent $1 \leq p < \infty$.

For $1 \leq p < \infty$, consider the Sobolev norm

$$\|q\|^p_{W^{k,p}(\mu_0,\ldots,\mu_k)} := \sum_{j=0}^{k} \|q^{(j)}\|^p_{L^p(\mu_j)},$$

where $\mu_0, \ldots, \mu_k$ are finite Borel measures with compact support in the real line.

For these Sobolev norms, if the multiplication operator

$$M\,q(z) = z\,q(z)\,,$$

is bounded as an operator on $\mathbb{P}^{k,2}$, the closure of the space of polynomials with respect to the norm $W^{k,2}(\mu_0, \ldots, \mu_k)$, the results of Lagomasino, Pijeira and Pérez Izquierdo show that the zeros of the Sobolev orthogonal polynomials are contained in the compact set $\{z : |z| \leq 2\|M\|\}$.

This fact allows to obtain the asymptotic behavior of the monic Sobolev orthogonal polynomials.

Under this hypothesis, they compute for every $j \geq 0$,

$$\lim_{n\to\infty} \|q_n^{(j)}\|_{S(\mu)}^{1/n}, \qquad \lim_{n\to\infty} \nu(q_n^{(j)}),$$

and

$$\lim_{n\to\infty} |q_n^{(j)}(z)|^{1/n}, \qquad \lim_{n\to\infty} \frac{q_n^{(j+1)}(z)}{nq_n^{(j)}(z)},$$

uniformly on compact subsets of $\{z : |z| > 2\|M\|\}$.

Also, they prove the following result.

Theorem (Lagomasino, Pijeira, Pérez Izquierdo)

Let $\mu_0, \dots, \mu_k$ *be finite Borel measures with compact support in the real line. If there exists a constant* C *with*

$$\mu_j \leq C \, \mu_{j-1}$$

for $1 \leq j \leq k$*, then the multiplication operator is bounded.*

Theorem (Álvarez, Pestana, R., Romera)

Let $\mu_0, \ldots, \mu_k$ *be finite Borel measures with compact support in the real line. Then the multiplication operator is bounded on* $W^{k,p}(\mu_0, \ldots, \mu_k)$ **if and only if** *the norms* $W^{k,p}(\mu_0, \ldots, \mu_k)$ *and* $W^{k,p}(\mu_0', \ldots, \mu_k')$ *are equivalent on the space of polynomials, where*

$$\begin{aligned} \mu_0' &:= \mu_0 + \mu_1 + \cdots + \mu_{k-1} + \mu_k \\ \mu_1' &:= \mu_1 + \cdots + \mu_{k-1} + \mu_k \\ &\ldots \\ \mu_{k-1}' &:= \mu_{k-1} + \mu_k \\ \mu_k' &:= \mu_k. \end{aligned}$$

Corollary (FAKE)

Let $\mu_0, \ldots, \mu_k$ be finite Borel measures with compact support in the real line. Then the multiplication operator is bounded on $X := W^{k,p}(\mu_0, \ldots, \mu_k)$.

PROOF:
Since $\mu_j \le \mu_j'$ for $0 \le j \le k$, we have directly

$$\|f\|_{W^{k,p}(\mu_0,\ldots,\mu_k)} \le \|f\|_{W^{k,p}(\mu_0',\ldots,\mu_k')}$$

for every $f \in X$, and so, the identity map

$$i : \left(X, \|\cdot\|_{W^{k,p}(\mu_0',\ldots,\mu_k')}\right) \to \left(X, \|\cdot\|_{W^{k,p}(\mu_0,\ldots,\mu_k)}\right)$$

is a linear continuous map.

By the open mapping theorem on Banach spaces i is also an open map, and since it is bijective, we have that

$$i^{-1} : \left(X, \|\cdot\|_{W^{k,p}(\mu_0,\dots,\mu_k)}\right) \to \left(X, \|\cdot\|_{W^{k,p}(\mu_0',\dots,\mu_k')}\right)$$

is also a linear continuous map. Therefore, there exists a positive constant C such that

$$\|f\|_{W^{k,p}(\mu_0',\dots,\mu_k')} \leq C \, \|f\|_{W^{k,p}(\mu_0,\dots,\mu_k)}$$

for every $f \in X$.

• V. Álvarez, D. Pestana, J. M. Rodríguez and E. Romera proved that, in many cases, the space of smooth functions (or polynomials) is dense in $W^{k,p}(\mu)$, with $1 \leq p < \infty$.

• V. Álvarez, D. Pestana, J. M. Rodríguez and E. Romera proved that, in many cases, the Sobolev space $W^{k,p}(\mu)$ is complete, with $1 \leq p < \infty$.

• J. M. Rodríguez and J. M. Sigarreta proved that, in every case, the Sobolev space $W^{k,p}(\mu)$ is complete for every $1 \leq p < \infty$.

• A. Portilla, Y. Quintana, J. M. Rodríguez and E. Tourís found the class of functions which can be approximated by smooth functions (or polynomials) in the norm of $W^{k,\infty}(w)$ (which is always a proper subset of $W^{k,\infty}(w)$).

Given $1 \leq p < \infty$, we say that a weight $w \in B_p(E)$ if $w_j^{-1} \in L^{1/(p-1)}(E)$.

Lemma (Álvarez, Pestana, R., Romera)

Let $\mu_0, \ldots, \mu_k$ be finite Borel measures on $[a, b]$ with $w_k \in B_p[a, b]$. If the support of μ_0 has at least k points, then there exists a constant C such that

$$\sum_{j=0}^{k} \|q^{(j)}\|_{L^p(\mu_j)}^p \leq C\|q\|_{L^p(\mu_0)}^p + C\|q^{(k)}\|_{L^p(\mu_k)}^p$$

for every polynomial q.

M. Castro and A. Durán proved that if the multiplication operator is bounded in $\mathbb{P}^{k,p}(\mu)$ then the support of each μ_j is a compact set.

We assume $\|f\|_{W^{k,p}(\mu)} < \infty$ for every polynomial f, and hence, for any $0 \leq j \leq k$,

$$\mu_j(\mathbb{R})^{1/p} = \|1\|_{L^p(\mu_j)} \leq \|z^j/j!\|_{W^{k,p}(\mu)} < \infty\,,$$

and consequently, each μ_j is finite.

By these facts, we just need to consider finite vectorial measures with compact support.

AN EXAMPLE:

$$d\mu_0 = dx \quad \text{on } [0,1], \qquad \mu_1 = \delta_0,$$

$$\|f\|^p_{W^{1,p}(\mu_0,\mu_1)} = \int_0^1 |f(x)|^p \, dx + |f'(0)|^p,$$

$$\mathbb{P}^{1,p}(\mu_0, \mu_1) = L^p[0,1] \times \mathbb{R}.$$

Each polynomial gives rise to an infinite number of equivalence classes in $\mathbb{P}^{1,p}(\mu_0, \mu_1)$.

This is not a rare example, since this type-Sobolev norm is considered, when $p = 2$, in the study of Sobolev Orthogonal Polynomials.

$P^{k,p}(\mu)$ versus $W^{k,p}(\mu)$:

• The definition of $\mathbb{P}^{k,p}(\mu)$ is simpler.

• With this "axiomatic" definition we do not know which are the elements of $\mathbb{P}^{k,p}(\mu)$.

• $\mathbb{P}^{k,p}(\mu)$ is not a space of functions if μ is not a "nice" measure.

AN EXAMPLE:

$$\mu_0 = 0, \qquad d\mu_1 = dx \quad \text{on } [0,1],$$

$$\|f\|^p_{W^{1,p}(\mu_0,\mu_1)} = \int_0^1 |f'(x)|^p \, dx,$$

$$[0] = \text{span}\,\{1\} = \{c : \; c \in \mathbb{R}\}.$$

Thus, the multiplication operator is not well defined, since

$$[0] = [1], \quad \text{i.e.} \quad \|0 - 1\|^p_{W^{1,p}(\mu_0,\mu_1)} = 0,$$

and

$$[0 \cdot x] \neq [1 \cdot x], \quad \text{i.e.} \quad \|0 - x\|^p_{W^{1,p}(\mu_0,\mu_1)} = \int_0^1 |-1|^p \, dx = 1 > 0.$$

HOW TO DEFINE REASONABLY WEIGHTED SOBOLEV SPACES?

Given $1 \leq p < \infty$ and a vectorial weight $w = (w_0, w_1, \dots, w_k)$ in $\mathbb{R}$, recall that we say that $w_j \in B_p(E)$ if $w_j^{-1} \in L^{1/(p-1)}(E)$.

Kufner and Opic define the open set

$$\begin{aligned} \Omega_j &:= \cup \left\{ I \text{ open interval } / \ w_j \in B_p(I) \right\} \\ &= \cup \left\{ I \text{ open interval } / \ w_j^{-1} \in L^{1/(p-1)}(I) \right\}. \end{aligned}$$

What role does this open set Ω_j play?

If $f^{(j)} \in L^p(w_j)$ and I is an interval with $w_j^{-1} \in L^{1/(p-1)}(I)$, then

$$\begin{aligned}\int_I |f^{(j)}| &= \int_I |f^{(j)}| w_j^{1/p} w_j^{-1/p} \\ &\leq \Big(\int_I |f^{(j)}|^p w_j \Big)^{1/p} \Big(\int_I w_j^{-1/(p-1)} \Big)^{(p-1)/p} < \infty \, ,\end{aligned}$$

and $f^{(j)} \in L^1(I)$ for any such interval; this guarantees that $f^{(j)}$ is locally integrable, which is a necessary condition for $f^{(j)}$ to be a weak derivative.

Consequently, $f^{(j-1)} \in AC(\Omega_j)$ if $1 \leq j \leq k$.

Definition

(Heuristic definition) Let us consider $1 \le p < \infty$ *and an appropriate vector measure* $\mu = (\mu_0, \dots, \mu_k)$ *on* $\mathbb{R}$*. We define the Sobolev space* $W^{k,p}(\mu) = W^{k,p}(\Delta, \mu)$*, with* $\Delta := \cup_{j=0}^{k} \operatorname{supp}(\mu_j)$*, as the space of equivalence classes of*

$$V^{k,p}(\mu) := V^{k,p}(\Delta, \mu)$$

$$:= \Big\{ f : \Delta \to \mathbb{R} \;\; : \;\; \|f\|_{W^{k,p}(\Delta,\mu)} := \Big(\sum_{j=0}^{k} \|f^{(j)}\|^p_{L^p(\Delta,\mu_j)} \Big)^{1/p} < \infty \,,$$

$$f^{(j)} \in AC_{loc}(\Omega_{j+1} \cup \cdots \cup \Omega_k) \text{ and } f^{(j)} \text{ satisfies}$$

$$\text{“pasting conditions” for } 0 \le j < k \Big\} \,,$$

with respect to the seminorm $\| \cdot \|_{W^{k,p}(\Delta,\mu)}$*.*

These pasting conditions are natural: a function must be as regular as possible. In a first step, we check if the functions and their derivatives are absolutely continuous up to the boundary (this fact holds in the following example), and then we join the contiguous intervals:

Example.

$\mu_0 := \delta_0$, $\mu_1 := 0$, $d\mu_2 := \chi_{[-1,0]}(x)dx$ and $d\mu_3 := \chi_{[0,1]}(x)dx$, where χ_A denotes the characteristic function of the set A.

Since $\Omega_1 = \emptyset$, $\Omega_2 = (-1, 0)$ and $\Omega_3 = (0, 1)$, $W^{3,p}(\mu)$ is the space of equivalence classes of

$$
\begin{aligned}
V^{3,p}(\mu) &= \Big\{ f \;:\; \|f\|_{W^{3,p}(\mu)} < \infty\,,\;\; f, f' \text{ satisfy “pasting conditions”}, \\
&\qquad\qquad\qquad f, f', \in AC((-1,0)) \text{ and } f, f', f'' \in AC((0,1)) \Big\} \\
&= \Big\{ f \;:\; \|f\|_{W^{3,p}(\mu)} < \infty\,,\;\; f, f' \text{ satisfy “pasting conditions”}, \\
&\qquad\qquad\qquad f, f', \in AC([-1,0]) \text{ and } f, f', f'' \in AC([0,1]) \Big\} \\
&= \Big\{ f \;:\; \|f\|_{W^{3,p}(\mu)} < \infty\,,\;\; f, f' \in AC([-1,1]) \text{ and } f'' \in AC([0,1]) \Big\}\,.
\end{aligned}
$$

In the current case, since f and f' are absolutely continuous in $[-1,0]$ and in $[0,1]$, we require that both are absolutely continuous in $[-1,1]$.

These heuristic concepts can be formalized as follows:

Definition

Let us consider $1 \le p < \infty$ *and* μ, ν *measures in* $[a, b]$. *We define*

$$\Lambda^{+}_{p,[a,b]}(\mu,\nu) := \sup_{a<x<b} \mu((a,x])\left\|(d\nu/ds)^{-1}\right\|_{L^{1/(p-1)}([x,b])},$$

$$\Lambda^{-}_{p,[a,b]}(\mu,\nu) := \sup_{a<x<b} \mu([x,b))\left\|(d\nu/ds)^{-1}\right\|_{L^{1/(p-1)}([a,x])},$$

where we use the convention $0 \cdot \infty = 0$.

Muckenhoupt inequality. *Let us consider* $1 \leq p < \infty$ *and* μ_0, μ_1 *measures in* $[a, b]$*. Then:*
(1) *There exists a real number c such that*

$$\left\| \int_x^b g(t)\, dt \right\|_{L^p((a,b],\mu_0)} \leq c \left\| g \right\|_{L^p((a,b],\mu_1)}$$

for any measurable function g in $[a, b]$*, if and only if* $\Lambda^+_{p,[a,b]}(\mu_0, \mu_1) < \infty$*.*
(2) *There exists a positive constant c such that*

$$\left\| \int_a^x g(t)\, dt \right\|_{L^p([a,b),\mu_0)} \leq c \left\| g \right\|_{L^p([a,b),\mu_1)}$$

for any measurable function g in $[a, b]$*, if and only if* $\Lambda^-_{p,[a,b]}(\mu_0, \mu_1) < \infty$*.*

Definition

Let us consider $1 \le p < \infty$. *A vector measure* $\overline{\mu} = (\overline{\mu}_0, \ldots, \overline{\mu}_k)$ *is a right completion of a vector measure* $\mu = (\mu_0, \ldots, \mu_k)$ *in* $\mathbb{R}$ *with respect to a in a right neighborhood* $[a, b]$, *if* $\overline{\mu}_k = \mu_k$ *in* $[a, b]$, $\overline{\mu}_j = \mu_j$ *in the complement of* $(a, b]$ *and*

$$\overline{\mu}_j = \mu_j + \tilde{\mu}_j \,, \qquad \text{in } (a, b] \quad \text{for } 0 \le j < k \,,$$

where $\tilde{\mu}_j$ *is any measure satisfying* $\tilde{\mu}_j((a, b]) < \infty$ *and* $\Lambda^+_{p,[a,b]}(\tilde{\mu}_j, \overline{\mu}_{j+1}) < \infty$.

Muckenhoupt inequality guarantees that if $f^{(j)} \in L^p(\mu_j)$ and $f^{(j+1)} \in L^p(\overline{\mu}_{j+1})$, then $f^{(j)} \in L^p(\overline{\mu}_j)$.

Remark

We can define a left completion of μ *with respect to a in a similar way.*

Definition

For $1 \leq p < \infty$ and a vector measure μ in $\mathbb{R}$, we say that a point a is right j-regular (respectively, left j-regular), if there exist a right completion $\overline{\mu}$ (respectively, left completion) of μ in $[a, b]$ and $j < i \leq k$ such that $\overline{w}_i \in B_p([a, b])$ (respectively, $B_p([b, a])$). Also, we say that a point $a \in \gamma$ is j-regular, if it is right and left j-regular.

Remark

1. *A point a is right j-regular (respectively, left j-regular), if at least one of the following properties holds:*

(a) *There exist a right (respectively, left) neighborhood $[a, b]$ (respectively, $[b, a]$) and $j < i \leq k$ such that $w_i \in B_p([a, b])$ (respectively, $B_p([b, a])$). Here we have chosen $\tilde{w}_j = 0$.*
(b) *There exist a right (respectively, left) neighborhood $[a, b]$ (respectively, $[b, a]$) and $j < i \leq k$, $\alpha > 0$, $\delta < (i - j)p - 1$, such that $w_i(x) \geq \alpha |x - a|^{\delta}$, for almost every $x \in [a, b]$ (respectively, $[b, a]$).*
2. *If a is right j-regular (respectively, left), then it is also right i-regular (respectively, left) for each $0 \leq i \leq j$.*

When we use this definition we think of a point $\{t\}$ as the union of two half-points $\{t^+\}$ and $\{t^-\}$. With this convention, each one of the following sets

$$(a,b) \cup (b,c) \cup \{b^+\} = (a,b) \cup [b^+,c) \neq (a,c)\,,$$
$$(a,b) \cup (b,c) \cup \{b^-\} = (a,b^-] \cup (b,c) \neq (a,c)\,,$$

has two connected components, and the set

$$(a,b) \cup (b,c) \cup \{b^-\} \cup \{b^+\} = (a,b) \cup (b,c) \cup \{b\} = (a,c)$$

is connected.

We use this convention in order to study the sets of continuity of functions: we want that if $f \in C(A)$ and $f \in C(B)$, where A and B are union of intervals, then $f \in C(A \cup B)$. With the usual definition of continuity, if $f \in C([a,b)) \cap C([b,c])$ then we do not have $f \in C([a,c])$. Of course, we have $f \in C([a,c])$ if and only if $f \in C([a,b^-]) \cap C([b^+,c])$, where by definition, $C([b^+,c]) = C([b,c])$ and $C([a,b^-]) = C([a,b])$. This idea can be formalized with a suitable topological space.

Let us introduce some more notation. We denote by $\Omega^{(j)}$ the set of j-regular points or half-points, i.e., $x \in \Omega^{(j)}$ if and only if x is j-regular, we say that $x^+ \in \Omega^{(j)}$ if and only if x is right j-regular, and we say that $x^- \in \Omega^{(j)}$ if and only if x is left j-regular. Obviously, $\Omega^{(k)} = \emptyset$ and $\Omega_{j+1} \cup \cdots \cup \Omega_k \subseteq \Omega^{(j)}$. Note that $\Omega^{(j)}$ depends on p.

Intuitively, $\Omega^{(j)}$ is the set of "good" points at the level j for the vector weight $(w_0, \ldots, w_k)$: every function f in the Sobolev space must verify that $f^{(j)}$ is continuous in $\Omega^{(j)}$.

Let us present now the class of measures that we use in the definition of Sobolev space.

Definition

We say that the vector measure $\mu = (\mu_0, \ldots, \mu_k)$ in $\mathbb{R}$ is p-admissible if μ_j is σ-finite and $\mu_j^*(\mathbb{R} \setminus \Omega^{(j)}) = 0$, for $1 \leq j < k$, and $\mu_k^* \equiv 0$, where $d\mu_j^* := d\mu_j - w_j \chi_{\Omega_j} dx$ and χ_A denotes the characteristic function of the set A (then $d\mu_k = w_k \chi_{\Omega_k} dx$).

Remark

1. *The hypothesis of p-admissibility is natural. It would not be reasonable to consider Dirac's deltas in* μ_j *in the points where* $f^{(j)}$ *is not continuous.*

2. *Note that there is not any restriction on* μ_0.

3. *Every absolutely continuous measure* $w = (w_0, \dots, w_k)$ *with* $w_j = 0$ *a.e. in* $\mathbb{R} \setminus \Omega_j$ *for every* $1 \leq j \leq k$, *is p-admissible (since then* $\mu_j^* = 0$*). It is possible to find a weight w which does not satisfy this condition, but it is a hard task.*

4. $(\mu_j)_s \leq \mu_j^*$, *and the equality usually holds.*

Definition

Let us consider $1 \leq p < \infty$ *and a p-admissible vector measure* $\mu = (\mu_0, \dots, \mu_k)$ *in* $\mathbb{R}$*. We define the Sobolev space* $W^{k,p}(\mu) = W^{k,p}(\Delta, \mu)$*, with* $\Delta := \cup_{j=0}^{k} \operatorname{supp}(\mu_j)$*, as the space of equivalence classes of*

$$V^{k,p}(\Delta, \mu) := \Big\{ f : \Delta \to \mathbb{R} \quad f^{(j)} \in AC_{loc}(\Omega^{(j)}) \text{ for } 0 \leq j < k \text{ and } \|f\|_{W^{k,p}(\Delta,\mu)} := \Big(\sum_{j=0}^{k} \|f^{(j)}\|_{L^p(\Delta,\mu_j)}^p \Big)^{1/p} < \infty \Big\},$$

with respect to the seminorm $\| \cdot \|_{W^{k,p}(\Delta,\mu)}$.

Theorem (Marcellán, Quintana, R.)

Let us consider the generalized Jacobi weight

$$w(x) := h(x)\,\Pi_{j=1}^{r}|x-c_j|^{\gamma_j}$$

on $[a,b]$ *with* $c_1,\dots,c_r \in \mathbb{R}$, $\gamma_1,\dots,\gamma_r \in \mathbb{R}$, $\gamma_j > -1$ *when* $c_j \in [a,b]$, *and* h *a measurable function satisfying* $0 < m \le h \le M$ *on* $[a,b]$ *for some constants* m, M.
Then we have

$$\|P'\|_{W^{k,2}(w,\lambda_1 w,\dots,\lambda_k w)} \le C_1\, n^2\, \|P\|_{W^{k,2}(w,\lambda_1 w,\dots,\lambda_k w)},$$
$$C_1 = C_1(a,b,c_1,\dots,c_r,\gamma_1,\dots,\gamma_r,m,M)$$

for every $\lambda_1,\dots,\lambda_k \ge 0$ *and* $P \in \mathbb{P}_n$.

INITIAL PAPERS:

J. M. Rodríguez, V. Álvarez, E. Romera, D. Pestana,
Generalized weighted Sobolev spaces and applications to Sobolev orthogonal polynomials I,
Acta Applicandae Mathematicae 80 (2004), 273-308.

J. M. Rodríguez, V. Álvarez, E. Romera, D. Pestana,
Generalized weighted Sobolev spaces and applications to Sobolev orthogonal polynomials II,
Approximation Theory and its Applications 18:2 (2002), 1-32.

V. Álvarez, D. Pestana, J. M. Rodríguez, E. Romera,
Weighted Sobolev spaces on curves,
Journal of Approximation Theory 119 (2002), 41-85.

IMPROVEMENTS OF THESE RESULTS:

J. M. Rodríguez,
Weierstrass' Theorem in weighted Sobolev spaces,
Journal of Approximation Theory 108 (2001), 119-160.

J. M. Rodríguez,
Approximation by polynomials and smooth functions in Sobolev spaces with respect to measures,
Journal of Approximation Theory 120 (2003), 185-216.

MULTIPLICATION OPERATOR:

J. M. Rodríguez,
The multiplication operator in Sobolev spaces with respect to measures,
Journal of Approximation Theory 109 (2001), 157-197.

APPROXIMATION ON INFINITY NORM:

A. Portilla, Y. Quintana, J. M. Rodríguez, E. Tourís,
Weierstrass' Theorem with weights,
Journal of Approximation Theory 127 (2004), 83-107.

A. Portilla, Y. Quintana, J. M. Rodríguez, E. Tourís,
Weighted Weierstrass' Theorem with first derivatives,
Journal of Mathematical Analysis and Applications 334 (2007),
1167-1198.

A. Portilla, Y. Quintana, J. M. Rodríguez, E. Tourís,
Weierstrass' Theorem in weighted Sobolev spaces with *k* derivatives,
Rocky Mountain Journal of Mathematics 37 (2007), 1989-2024.

KOLMOGOROV-SZEGÖ-KREIN THEOREM:

J. M. Rodríguez, D. V. Yakubovich,
A Kolmogorov-Szegö-Krein type condition for weighted Sobolev spaces,
Indiana University Mathematics Journal 54 (2005), 575-598.

SIMPLIFIED VERSION:

J. M. Rodríguez,
A simple characterization of weighted Sobolev spaces with bounded multiplication operator,
Journal of Approximation Theory 153 (2008), 53-72.

J. M. Rodríguez, J. M. Sigarreta,
Sobolev spaces with respect to measures in curves and zeros of Sobolev orthogonal polynomials,
Acta Applicandae Mathematicae 104 (2008), 325-353.

NON-DIAGONAL SOBOLEV NORMS:

A. Portilla, J. M. Rodríguez, E. Tourís,
The multiplication operator, zero location and asymptotic for non-diagonal Sobolev norms,
Acta Applicandae Mathematicae 111 (2010), 205-218.

A. Portilla, Y. Quintana, J. M. Rodríguez, E. Tourís,
Zero location and asymptotic behavior for extremal polynomials with non-diagonal Sobolev norms,
Journal of Approximation Theory 162 (2010), 2225-2242.

Y. Quintana, J. M. Rodríguez,
Measurable diagonalization of positive definite matrices,
Journal of Approximation Theory 185 (2014), 91-97.

OPTIMIZATION OF MUCKENHOUPT INEQUALITY:

E. Colorado, D. Pestana, J. M. Rodríguez, E. Romera,
Muckenhoupt inequality with three measures and Sobolev orthogonal polynomials,
Journal of Mathematical Analysis and Applications 407 (2013), 369-386.

J. M. Rodríguez,
Zeros of Sobolev orthogonal polynomials via Muckenhoupt inequality with three measures,
Acta Applicandae Mathematicae 142:1 (2016), 9-37.

MARKOV-TYPE INEQUALITIES:

F. Marcellán, Y. Quintana, J. M. Rodríguez,
Weighted Sobolev spaces: Markov-type inequalities and duality,
Bulletin of Mathematical Sciences (2018) 8:233-256.

F. Marcellán, J. M. Rodríguez,
Lupas-type inequality and applications to Markov-type inequalities in weighted Sobolev spaces,
Bulletin of Mathematical Sciences Vol. 11, No. 1 (2021) 1950022

THANK YOU FOR YOUR ATTENTION