Seminario Iberoamericano de Análisis Matemático y Matemática Aplicada

Ortogonalidad y estabilidad de sistemas lineales: algunas aplicaciones

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Linear systems and stability

Continuous-time linear systems

Given a linear system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t), \quad \mathbf{x}(t) \in \mathbb{R}^n, \quad \mathbf{A} \in \mathbb{R}^{\mathbf{n}\times\mathbf{n}},$$

the solutions have the form

$$\mathbf{x} = \sum_i \mathbf{v}_i e^{\lambda_i t},$$

where $\mathbf{v}_i$ is a vector with entries independent of t, and λ_i is an eigenvalue of $\mathbf{A}$, that is

$$\det(\mathbf{A} - \lambda_i \mathbf{I}) = 0.$$

Definition (Stability of linear systems)

The linear system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t),$$

is asymptotically stable if $\lim_{t\to\infty} \mathbf{x} = \mathbf{0}$.

Hurwitz polynomials

Theorem (Stability of continuous-time linear systems)

The linear system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$$

is asymptotically stable if and only if all eigenvalues of $\mathbf{A}$ have strictly negative real part, that is, if the zeros of the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$ are located in the left half-plane.

Definition

A polynomial $f \in \mathbb{R}[x]$ is said to be a Hurwitz polynomial if its zeros belong to the set $\mathbb{C}^- = \{a + ib : a, b \in \mathbb{R} \text{ and } a < 0\}$.

Discrete-time linear systems

Consider the linear system

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k, \quad k \geq 0,$$

then we have $\mathbf{x}_k = \mathbf{A}^k \mathbf{x}_0$. The solutions have the form

$$\mathbf{x}_k = \sum_i \alpha_i \lambda_i^k \mathbf{v}_i,$$

where $\mathbf{v}$ are the eigenvectors, α_i depend on the initial condition $\mathbf{x}_0$, and λ_i are the eigenvalues of $\mathbf{A}$, that is

$$\det(\mathbf{A} - \lambda_i \mathbf{I}) = 0.$$

Definition (Stability of linear systems)

The linear system

$$\mathbf{x}_{k+1} = \mathbf{A}\mathbf{x}_k, \quad k \geq 0,$$

is asymptotically stable if $\lim_{k\to\infty} \mathbf{x_k} = \mathbf{0}$.

Schur polynomials

Theorem (Stability of discrete-time linear systems)

The linear system

$$\dot{\mathbf{x}}_{k+1} = \mathbf{A}\mathbf{x}_k, \quad k \geq 0,$$

is asymptotically stable if and only if all eigenvalues of $\mathbf{A}$ satisfy $|\lambda_i| < 1$, that is, if the zeros of the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$ are located in the unit disc.

Definition

A polynomial $f \in \mathbb{R}[z]$ is said to be a Schur polynomial if its zeros belong to the set $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Stable and orthogonal polynomials

Stable polynomials

Problems

- To characterize stability.
- To determine if a given polynomial is stable (Hurwitz or Schur) without computing explicitly its zeros.

Criteria for Hurwitz polynomials (among others):

- Routh-Hurwitz
- Stability test
- Hermite-Biehler theorem
- Markov parameters

Routh-Hurwitz test

Definition

Consider the real polynomial

$$f(t) = a_n t^n + a_{n-1} t^{n-1} + \ldots + a_1 t + a_0.$$

Denote by $H(f)$ the $n \times n$ Hurwitz matrix of f, defined by

$$\begin{pmatrix} a_{n-1} & a_{n-3} & a_{n-5} & \ldots & 0 & 0 \\ a_n & a_{n-2} & a_{n-4} & \ldots & 0 & 0 \\ 0 & a_{n-1} & a_{n-3} & \ldots & 0 & 0 \\ 0 & a_n & a_{n-2} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & a_1 & 0 \\ 0 & 0 & 0 & \ldots & a_2 & a_0 \end{pmatrix}. \tag{1}$$

Routh-Hurwitz test (cont.)

Denote by Δ_i $(i = 1, \ldots, n)$ the principal minors of $H(f)$:

$$\Delta_1 = a_{n-1}, \Delta_2 = \begin{pmatrix} a_{n-1} & a_{n-3} \\ a_n & a_{n-2} \end{pmatrix}, \Delta_3 = \begin{pmatrix} a_{n-1} & a_{n-3} & a_{n-5} \\ a_n & a_{n-2} & a_{n-4} \\ 0 & a_{n-1} & a_{n-3} \end{pmatrix}, \ldots$$

Theorem (Routh-Hurwitz)

Let $f(t) = a_n t^n + a_{n-1} t^{n-1} + \ldots + a_1 t + a_0 \in \mathbb{R}[t]$ with $a_n > 0$ and $H(f)$ the Hurwitz matrix associated with f. Then $f(t)$ is a Hurwitz polynomial if and only if $\Delta_1 > 0, \Delta_2 > 0, \ldots, \Delta_n > 0$.

Stability test

Theorem (Stability test)

Given the real polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, with $a_{n-1} \neq 0$, define the polynomial

$$Q(x) = a_{n-1}x^{n-1} + (a_{n-2} - \frac{a_n}{a_{n-1}}a_{n-3})x^{n-2} + a_{n-3}x^{n-3} +$$

$$+(a_{n-4} - \frac{a_n}{a_{n-1}}a_{n-5})x^{n-4} + \ldots$$

If $P(x)$ have positive coefficients, then $P(x)$ is a Hurwitz polynomial if and only if $Q(x)$ is a Hurwitz polynomial.

Hermite-Biehler theorem

Definition

The polynomials $h(u)$ and $g(u)$ of degree m (or $\deg h = m$ and $\deg g = m-1$) are a positive pair if their zeros $u_1, u_2, \ldots, u_m$ and $u'_1, u'_2, \ldots, u'_m$ (or $u'_1, u'_2, \ldots, u'_{m-1}$) are simple, negative, satisfy the interlacing property

$$u'_1 < u_1 < u'_2 < u_2 < \ldots < u'_m < u_m < 0$$

$$(\textit{or } u_1 < u'_1 < u_2 < u'_2 < \ldots < u'_{m-1} < u_m < 0),$$

and their leading coefficients have the same sign.

Theorem (Hermite-Biehler)

The real polynomial $f(x) = h(x^2) + xg(x^2)$ is a Hurwitz polynomial if and only if $h(x)$ and $g(x)$ are a positive pair.

Markov's parameters

Definition (Markov's parameters)

Let $f(x) = h(x^2) + xg(x^2)$ *be a real polynomial of degree* m*. Consider*

$$\frac{g(x)}{h(x)} = \frac{\tilde{s}_0}{x} - \frac{\tilde{s}_1}{x^2} + \frac{\tilde{s}_2}{x^3} - \ldots + \frac{\tilde{s}_{2n-2}}{x^{2n-1}} - \frac{\tilde{s}_{2n-1}}{x^{2n}} + \ldots, \qquad (2)$$

$$\frac{g(x)}{h(x)} = \tilde{s}_{-1} + \frac{\tilde{s}_0}{x} - \frac{\tilde{s}_1}{x^2} + \frac{\tilde{s}_2}{x^3} - \ldots + \frac{\tilde{s}_{2n-2}}{x^{2n-1}} - \frac{\tilde{s}_{2n-1}}{x^{2n}} + \ldots, \qquad (3)$$

for the even ($m = 2n$) *and odd* ($m = 2n+1$) *cases, respectively. The numbers*

$$[\tilde{s}_0, \ldots, \tilde{s}_{2n-1}] \quad \textit{and} \quad [\tilde{s}_{-1}, \tilde{s}_0, \ldots, \tilde{s}_{2n-1}]$$

are known as Markov's parameters.

Theorem (Markov's parameters)

The polynomial $f(x)$ is a Hurwitz polynomial if and only if the Hankel matrices

$$\widetilde{H}_{n-1} = \begin{pmatrix} \tilde{s}_0 & \tilde{s}_1 & \dots & \tilde{s}_{n-1} \\ \tilde{s}_1 & \tilde{s}_2 & \dots & \tilde{s}_n \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{s}_{n-1} & \tilde{s}_n & \dots & \tilde{s}_{2n-2} \end{pmatrix}, \widetilde{H}^{(1)}_{n-1} = \begin{pmatrix} \tilde{s}_1 & \tilde{s}_2 & \dots & \tilde{s}_n \\ \tilde{s}_2 & \tilde{s}_3 & \dots & \tilde{s}_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{s}_n & \tilde{s}_{n+1} & \dots & \tilde{s}_{2n-1} \end{pmatrix} \tag{4}$$

are positive definite. For the odd case, it is also required $\tilde{s}_{-1} > 0$.

Criteria for Schur polynomials

Consider the polynomial (real coefficients)

$$S_n(z) = z^n + p_{n-1}z^{n-1} + \ldots + p_1 z + p_0,$$

- A necessary condition is $|p_0| < 1$.
- Hermite-Biehler type theorems
- Jury test

Hermite-Biehler type theorems

Consider the polynomial (real coefficients)

$$S_n(z) = z^n + p_{n-1}z^{n-1} + \ldots + p_1 z + p_0,$$

and define

$$R(\theta) = cos(n\theta) + p_{n-1}cos((n-1)\theta) + \ldots + cos(\theta)p_1 + p_0,$$

$$I(\theta) = sin(n\theta) + p_{n-1}sin((n-1)\theta) + \ldots + sin(\theta)p_1,$$

i.e. the real and imaginary parts of $S_n(z)$.

Theorem

$S_n(z)$ (with $|p_0| < 1$) is a Schur polynomial if and only if

(i) $R(\theta)$ has exactly n zeros in $[0, \pi]$,

(ii) $I(\theta)$ has exactly $n + 1$ zeros in $[0, \pi]$,

(iii) The zeros of $R(\theta)$ and $I(\theta)$ interlace.

Hermite-Biehler type theorems (cont.)

Let $S_n(z) = S_n^s(z) + S_n^a(z)$, where

$$S_n^s(z) = \frac{1}{2}[S_n(z) + z^n S_n(\frac{1}{z})], \quad S_n^a(z) = \frac{1}{2}[S_n(z) - z^n S_n(\frac{1}{z})]. \tag{5}$$

Theorem

$S_n(z)$ is a Schur polynomial if and only if

(i) $S_n^s(z)$ and $S_n^a(z)$ are polynomials of degree n with coefficients of the same sign.

(ii) $S_n^s(z)$ and $S_n^a(z)$ have simple and interlaced zeros on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

Remark: The polynomial $z^n S_n(\frac{1}{z})$ is usually called the *reciprocal (or reversed) polynomial* of $S_n(z)$ and it is denoted by $S_n^*(z)$. We say that a polynomial $A(z)$ is symmetric if $A(z) = A^*(z)$ and anti-symmetric if $A(z) = -A^*(z)$. This explains the notation in (5) since $S_n^s(z)$ and $S_n^a(z)$ are clearly symmetric and anti-symmetric polynomials, respectively.

Jury test

Algorithm 1: Algorithm to determine if a given polynomial is Schur

Input: Any monic polynomial $S_n(z) = z^n + p_{n-1}z^{n-1} + \ldots + p_0$ (coefficients can be complex).

Output: Determination of the Schur character of $S_n(z)$.

1 initialization;

2 **for** $i = 0, 2, \ldots n-1$ **do**

3 $S_n^{(i=0)} = S_n(z)$,

4 Verify $|p_0^{(i)}| < 1$,

5 Compute $S_n^{(i+1)}(z) = \frac{1}{z}[\frac{S_n^{(i)}(z) - S_n^{(i)}(0)z^{n-i}\overline{S_n^{(i)}(\frac{1}{\bar{z}})}}{(1-|S_n^{(i)}(0)|^2)}]$,

6 Return to step 4 until the condition is not satisfied. In such a case, $S_n(z)$ is not a Schur polynomial. If the condition in step 4 holds for $p_0^{(0)}, p_0^{(1)}, ..., p_0^{(n-1)}$, then $S(z)$ is a Schur polynomial.

7 **return**

OPRL vs OPUC

Let μ and σ be positive measures supported on $E \in \mathbb{R}$ and $\mathbb{T}$, respectively. Then, there exist (monic) sequences $\{P_n\}_{n\geq 0}$ and $\{\Phi_n\}_{n\geq 0}$ such that

$$\int_E P_n(x)P_m(x)d\mu(x) = \gamma_n\delta_{n,m}, \quad \gamma_n > 0, \quad n,m \geq 0.$$

$$\int_{\mathbb{T}} \Phi_n(z)\overline{\Phi_m(z)}d\sigma(z) = k_n\delta_{n,m}, \quad k_n > 0, \quad n,m \geq 0.$$

The moments are defined by

$$s_n = \int_E x^n d\mu(x), \quad n \geq 0, \quad c_n = \int_{\mathbb{T}} z^n d\sigma(z), \quad n \in \mathbb{Z}. \qquad (6)$$

The corresponding Gram matrices are Hankel and Toeplitz matrices, respectively.

* Existence of $\{P_n\}_{n\geq 0}$ and $\{\Phi_n\}_{n\geq 0}$ if and only if positivity of the Gram matrices.

Second kind polynomials and zeros

- The zeros of P_n are simple, real, and located in E.
- The zeros of P_n and P_{n-1} interlace.
- The zeros of Φ_n are located in the unit disc $\mathbb{D}$.

The second kind polynomial associated with P_n is defined by

$$Q_n(x) = \int_E \frac{P_n(z) - P_n(x)}{z - x} d\mu(z). \tag{7}$$

Q_n is a polynomial of degree $n - 1$. Its zeros interlace with the zeros of P_n.

Recurrence relations

For OPRL, we have the TTRR

$$P_{n+1}(x) = (x - b_n)P_n(x) - a_n P_{n-1}(x), \quad n = 0, 1, 2, ..., \tag{8}$$
$$Q_{n+1}(x) = (x - b_n)Q_n(x) - a_n Q_{n-1}(x), \quad n = 1, 2, ..., \tag{9}$$

where $P_{-1}(x) := 0$, $P_0(x) = 1$, $Q_0(x) = 0$, $Q_1(x) = \phi$.

- $\{a_n\}_{n\geq 1}$ are positive and $\{b_n\}_{n\geq 0}$ are real.

For OPUC, we have the Szegő recurrence

$$\Phi_{n+1}(z) = z\Phi_n(z) + \Phi_{n+1}(0)\Phi_n^*(z), \tag{10}$$

- $\{\Phi_{n+1}(0)\}_{n\geq 0}$ are complex numbers in $\mathbb{D}$ (Verblunsky, Schur, reflection coefficients).
- $\Phi_n^*(z)$ is the reciprocal polynomial as defined before.
- Jury test is equivalent to Szegő recurrence.
- $\Phi_n(z)$ is a Schur polynomial.

Inverse theorems

- Given arbitrary sequences $\{a_n\}_{n\geq 1}$ (positive) and $\{b_n\}_{n\geq 0}$ (real), the TTRR defines an orthogonal sequence (Favard's theorem).
- Given any polynomials P_n and P_{n-1} with real, simple and interlaced zeros, there exists a sequence $\{P_0, P_1, \ldots, P_{n-1}, P_n, \ldots\}$ that is orthogonal with respect to some measure μ (Geronimus-Wendroff Theorem).
- Given an arbitrary sequence of numbers $\{a_n\}_{n\geq 0}$ in $\mathbb{D}$, Szegő recursion defines an orthogonal sequence with respect to some measure σ (Verblunsky's theorem).
- Given any polynomial Φ_n with zeros in $\mathbb{D}$, there exists a sequence $\{\Phi_0, \Phi_1, \ldots, \Phi_n, \ldots\}$ orthogonal with respect to some measure σ ("Geronimus-Wendroff theorem for OPUC").

*** The last statement implies Schur is *equivalent* to OPUC.

Relation between orthogonality and stability

Background

- Connections with the moment problem and continued fractions (Gantmacher, 1959)
- Euclid's algorithm, orthogonal polynomials and Routh-Hurwitz algorithm (Genin, 1991)
- Matrix case (Choque, 2015)

Hurwitz → OPRL (even degree)

Theorem (Hurwitz → OPRL)

Let $f(x)$ be a monic Hurwitz polynomial and define h and g such that $f(x) = h(x^2) + xg(x^2)$. If $f(x) = x^{2n} + \sigma_1 x^{2n-1} + \ldots + \sigma_{2n-1}x + \sigma_{2n}$ and

$$\begin{aligned} P_n(x) &= \sum_{i=0}^{n} (-1)^i \sigma_{2i} x^{n-i} = (-1)^n h(-x), \quad \text{with} \quad \sigma_0 = 1, \\ Q_n(x) &= \sum_{i=0}^{n-1} (-1)^i \sigma_{2i+1} x^{n-1-i} = (-1)^{n-1} g(-x), \end{aligned}$$

then P_n has degree n and it is orthogonal with respect to some meaure μ supported on $[0, \infty)$ whose moments $\{s_0, \ldots, s_{2n-1}\}$ are the Markov's parameters associated with f, and Q_n is the corresponding second kind polynomial.[1]

[1] Martínez, N.; Garza, L.E.; Aguirre-Hernández, B. On sequences of Hurwitz polynomials related to orthogonal polynomials. *Linear Multilinear A.*, **2019**, 67(11), 2191–2208.

Hurwitz → OPRL (odd degree)

Theorem (Hurwitz → OPRL)

Consider $f(x) = x^{2n+1} + \sigma_1 x^{2n} + \ldots + \sigma_{2n} x + \sigma_{2n+1}$ and define

$$\begin{aligned} \sigma_1 P_n(x) &= \sum_{i=0}^{n} (-1)^i \sigma_{2i+1} x^{n-i} = (-1)^n h(-x), \\ s_{-1} P_n(x) - Q_n(x) &= \sum_{i=0}^{n} (-1)^i \sigma_{2i} x^{n-i} = (-1)^n g(-x), \end{aligned}$$

with $\sigma_0 = 1$ y $\sigma_1 = 1/s_{-1}$, then P_n is an OPRL with respect to some measure μ whose moments $\{s_0, \ldots, s_{2n-1}\}$ are the Markov's parameters associated with f, and Q_n is the corresponding second kind polynomial.[2]

[2] Martínez, N.; Garza, L.E.; Aguirre-Hernández, B. On sequences of Hurwitz polynomials related to orthogonal polynomials. *Linear Multilinear A.*, **2019**, 67(11), 2191–2208.

OPRL → Hurwitz (even degree)

Theorem (OPRL → Hurwitz)

Let $\{P_n\}_{n\geq 0}$ be an OPRL with respect to some μ supported on $[0,\infty)$. Let $\{Q_n\}_{n\geq 0}$ be the second kind polynomials. Then, [3]

$$f_{2n}(x) = (-1)^n P_n(-x^2) + (-1)^{n-1} x Q_n(-x^2), \quad n \geq 1, \tag{11}$$

is a monic Hurwitz polynomial of degree $2n$.

[3] Martínez, N.; Garza, L.E.; Aguirre-Hernández, B. On sequences of Hurwitz polynomials related to orthogonal polynomials. *Linear Multilinear A.*, **2019**, 67(11), 2191–2208.

OPRL $\to$ Hurwitz (odd degree)

Theorem (OPRL $\to$ Hurwitz)

Let $\{P_n\}_{n\geq 0}$ be an OPRL with respect to a positive measure μ supported in $[0,\infty)$, and let $\{Q_n\}_{n\geq 0}$ be the second kind polynomials. Define $G_n(x) = Q_n(x) + \beta_n P_n(x)$ with $\beta_n > 0$ such that the zeros of P_n and G_n are positive and interlaced. Then,[4]

$$f_{2n+1}(x) = (-1)^n x P_n(-x^2) + (-1)^n G_n(-x^2), \quad n \geq 0, \tag{12}$$

is a monic Hurwitz polynomial of degree $2n+1$.

Remark: β_n is a free parameter that varies with the degree of P_n.

[4] Martínez, N.; Garza, L.E.; Aguirre-Hernández, B. On sequences of Hurwitz polynomials related to orthogonal polynomials. *Linear Multilinear A.*, **2019**, 67(11), 2191–2208.

Sequences of Hurwitz polynomials via OPRL

Is there some β (independent of the degree) for the previous theorem?

Proposition

Let $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ be as before. Then, the sequence $\{t_n\}_{n\geq 1}$ with

$$t_n = -\frac{Q_n(0)}{P_n(0)},$$

is strictly increasing.

* If $t_n \to M > 0$, it suffices to take $\beta > M$.

A Hurwitz sequence satisfying a TTRR

Theorem (Recurrence relation for Hurwitz polynomials)

If there exists a finite M such that $-\frac{Q_n(0)}{P_n(0)} \leq M$ for every n, then the sequence $\{f_n\}_{n\geq 0}$ of Hurwitz polynomial constructed from $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ with $\beta > M$ satisfy

$$f_n(x) = (x^2 + b_{[\frac{n}{2}]-1})f_{n-2}(x) - a_{[\frac{n}{2}]-1}f_{n-4}(x), \tag{13}$$

where $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ are the coefficients on the TTRR of (8) and (9), with initial conditions. $a_0 f_{-2}(x) := -\phi x$, $a_0 f_{-1}(x) := \phi$, $f_0(x) = 1$, $f_1(x) = x + \beta$.

Remark: Given arbitrary (some restrictions apply) sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 0}$, the polynomials constructed via (13) are Hurwitz (Favard-type theorem for Hurwitz polynomials).

Conclusions and further remarks

- To any Hurwitz polynomial, we can associate a (finite) sequence $\{P_k\}_{k=0}^{n}$.
- The even and odd part of a Hurwitz polynomial are orthogonal and second kind polynomials, respectively.
- A sequence of Hurwitz polynomials can be constructed by using OPRL (and their second kind polynomials) when the measure is supported on $[0, \infty)$.
- This sequence satisfies a TTRR and a Favard's theorem.
- Schur polynomials and OPUC are *equivalent*.
- By use of a Möbius transformation $z = \frac{x+1}{x-1}$, mapping $\mathbb{D}$ onto $\mathbb{C}^-$, a polynomial $S_n(z)$ can be mapped into a polynomial $f_n(x)$ by using

$$(x-1)^n S_n\left(\frac{x+1}{x-1}\right) = f_n(x). \tag{14}$$

  $S_n(z)$ is a Schur polynomials if and only if $f_n(x)$ is a Hurwitz polynomial.

Applications

Assume the matrix $\mathbf{A}$ on the linear system depends on a vector of d parameters $\mathbf{q} \in D \subset \mathbb{R}^d$,i.e.

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{q})\mathbf{x}(t).$$

Then, the characteristic polynomial $p(\lambda, \mathbf{q}) = \det(\mathbf{A}(\mathbf{q}) - \lambda I)$ also depends on the parameters.

The above linear system is said to be robustly stable if $p(\lambda, \mathbf{q})$ is a Hurwitz polynomial for every $\mathbf{q} \in D$.

$\mathbf{q}$ is called an uncertainty parameter, and several types are considered.

Example: Interval uncertainty

Consider the family of polynomials

$$f(t, \mathbf{q}) = q_0 + q_1 x + \cdots + q_{n-1} x^{n-1} + q_n x^n,$$

where the coefficients satisfy

$$q_0 \in [x_0, y_0], q_1 \in [x_1, y_1], \cdots q_n \in [x_n, y_n].$$

In this case, $\mathbf{q} = (q_0, \ldots, q_n)^t \in \mathbb{R}^{n+1}$ and $\mathbf{q} \in D \subset \mathbb{R}^{n+1}$ where D is a hypercube.

Teorem (Kharitonov)

Every polynomial in the family is Hurwitz if and only if the following polynomials are Hurwitz:

$$\begin{aligned}
K^1(x) &= x_0 + x_1 x + y_2 x^2 + y_3 x^3 + x_4 x^4 + x_5 x^5 \cdots, \\
K^2(x) &= x_0 + y_1 x + y_2 x^2 + x_3 x^3 + x_4 x^4 + y_5 x^5 \cdots, \\
K^3(x) &= y_0 + x_1 x + x_2 x^2 + y_3 x^3 + y_4 x^4 + x_5 x^5 \cdots, \\
K^4(x) &= y_0 + y_1 x + x_2 x^2 + x_3 x^3 + y_4 x^4 + y_5 x^5 \cdots.
\end{aligned}$$

More general types of uncertainty

- Affine: coefficients are linear combinations of q_i.

$$p(x, \mathbf{q}) = x^2 + (3q_1 - q_2)x + 2q_0 + 3q_1$$

- Multilinear: coefficients are linear in q_i with the other q_j $(j \neq i)$ fixed.

$$p(x, \mathbf{q}) = x^2 + (3q_1 q_2 + q_0)x + 2q_0 q_1$$

- Polynomic: coefficients are multivariate polynomials in the parameters q_i.

$$p(x, \mathbf{q}) = x^2 + (3q_1^2 q_2 + q_0)x + 2q_0^3 q_1$$

Robust stability via classical OPRL

Idea: Introduce a parameter on the orthogonality measure:

$$\mu(x) = x^{\alpha} e^{-x} \Rightarrow \mu(x,t) = x^{\alpha} e^{-tx},$$

where t is a positive parameter.

- $\{P_n(x,t)\}_{n\geq 0}$ (and $\{Q_n(x,t)\}_{n\geq 0}$) will be orthogonal (and second kind) for every $t > 0$.
- A sequence of Hurwitz polynomials $\{F_n(x,t)\}_{n\geq 0}$ can be constructed.
- Such a sequence will be robustly stable for every $t > 0$.

Example ($\alpha = 0$): $P_3(x,t) = x^3 - 9tx^2 + 18t^2x - 6t^3$ (Polynomic uncertainty)

Robust stability via classical OPRL (cont.)

- Other measures (e.g. Jacobi) can be used.[5]
- More parameters can be introduced.
- Behavior of the zeros (with respect to t) was analyzed.
- Different uncertainty structures can be obtained.
- Applications in control design (pole placement).

[5] Arceo, A., Garza, L.E., & Romero, G. Robust Stability of Hurwitz Polynomials Associated with Modified Classical Weights. *Mathematics*, **2019**, 7, 818.

2. New stability criteria for Schur polynomials

By using Möbius transformations, we can study Schur stability via Hurwitz stability. An alternative for this is the use of the Szegő transformation:

The mappings $z = e^{i\theta} \mapsto 2\cos(\theta)$, with $\theta \in [0, 2\pi)$ and $x \mapsto \arccos(x/2)$ define a two-one correspondence between $\mathbb{T}$ and $[-2, 2]$. More precisely, we say $d\mu = Sz(d\sigma)$ if and only if $d\sigma(\theta) = d\sigma(-\theta)$ and

$$\int_0^{2\pi} f(\theta) d\sigma(\theta) = \int_{-2}^{2} f(arcos(x/2)) d\mu(x),$$

for any function f such that $f(\theta) = f(-\theta)$.

New stability criteria for Schur polynomials (cont.)

- The relation between the OP families $\{P_n\}_{n\geq 0}$ and $\{\Phi_n\}_{n\geq 0}$ is known.
- By using this transformation and the relations Hurwitz-OPRL and Schur-OPUC, a new interlacing criterion for Schur polynomials has been obtained. [6]
- This improves other known interlacing criteria since it is required to verify the interlacing of polynomials of degree $n/2$.

[6] Garza, L.E., Martínez, N. & Romero, G., New stability criteria for discrete linear systems based on orthogonal polynomials. *Mathematics*, **2020**, 8, 1322.

3. Robust stabilization of interval plants with uncertain time-delay

Let us consider an interval plant with uncertain time-delay, represented by

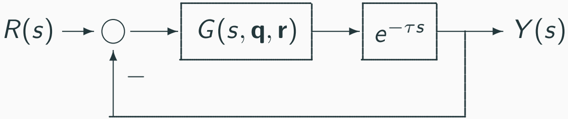

Figure 1: Closed-loop system for the interval plant with time-delay.

Here, $G(s,\mathbf{q},\mathbf{r}) = \eta(s,\mathbf{q})/\delta(s,\mathbf{r})$ where both $\eta(s,\mathbf{q})$ and $\delta(s,\mathbf{r})$ are polynomials with interval uncertainty and $e^{-\tau s}$ represents the time-delay.

- The system is robustly stable if and only if the quasi-polynomial

$$p(s,\mathbf{q},\mathbf{r},\tau) = \delta(s,\mathbf{r}) + e^{-\tau s}\eta(s,\mathbf{q})$$

satisfies $p(j\omega,\mathbf{q},\mathbf{r},\tau) \neq 0$ for $\omega \geq 0$, $\tau \geq 0$, $\mathbf{q} \in Q$, $\mathbf{r} \in R$.

The value set (geometric criterion)

The four Kharitonov polynomials are used to compute the vertices of a rectangle that determines the stability of a given polynomial with interval uncertainty.

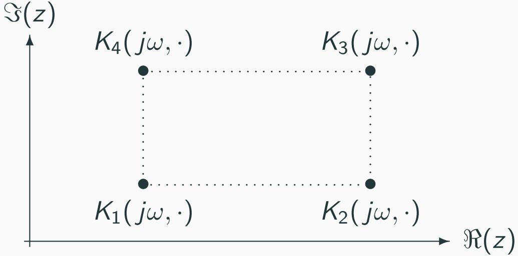

Figure 2: Value set associated with an interval polynomial family.

The value set of $p(s, \mathbf{q}, \mathbf{r}, \tau) = \delta(s, \mathbf{r}) + e^{-\tau s}\eta(s, \mathbf{q})$ will be a geometric shape obtained by the sum of a rectangle with a rotated rectangle.

Compensator for unstable systems

If a system is not stable, then a compensator $C(s) = B(s)/A(s)$ can be introduced:

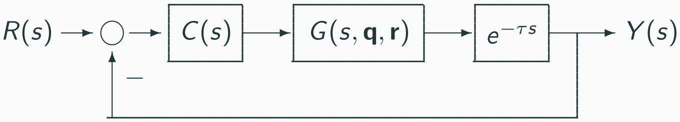

Figure 3: Closed-loop system for the interval plant with time-delay.

Idea: Choose $C(s)$ so that $A(s)$ (resp. $B(s)$) is an orthogonal (second kind) polynomial satisfying some inequality to guarantee the stability. [7]

[7] Zamora, P. Arceo, A., Martínez, N., Garza, L.E., & Romero, G. Robust Stabilization of interval plants with uncertain time-delay using the value set concept. *Mathematics*, **2021**, 9, 429.

Example

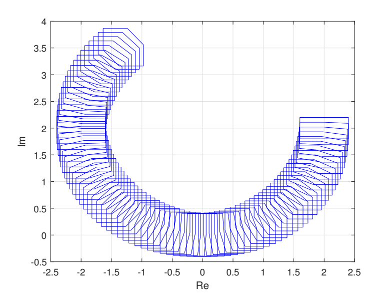

Figure 4: Value set for an unstable plant.

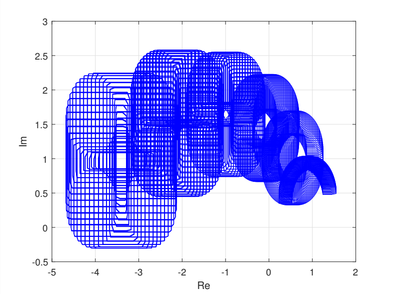

Figure 5: Value set for a stabilized plant.

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Future (and current) work

- Construct robustly stable families via recurrence relations
- Obtain analogues for discrete linear systems (Schur polynomials)
- Consider other applications

References

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Thanks!