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A friendly approach to Sobolev orthogonal polynomials

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The role of standard orthogonal polynomials

Let μ be a positive Borel measure supported in a subset E of the real line. Determine a polynomial Q_{n-1} of degree $\leq n-1$ that minimizes

$$\int_E |x^n - \pi_{n-1}(x)|^2 d\mu(x).$$

among all the polynomials π_{n-1} of degree $\leq n-1$. This is equivalent to find the n-th monic polynomial orthogonal $P_n = x^n - Q_{n-1}(x)$ with respect to the inner product

$$\langle f, g \rangle = \int_E f(x) g(x) d\mu(x). \tag{1}$$

The role of standard orthogonal polynomials

These polynomials satisfy a three term recurrence relation (TTRR)

$$xP_n(x) = P_{n+1}(x) + b_nP_n(x) + a_n^2P_{n-1}(x), n \geq 0, \tag{2}$$

with initial conditions $P_{-1}(x) = 0$ and $P_0(x) = 1$.
The connection with Jacobi matrices and discretization of Schrödinger operators

$$(Ju)_n = a_{n+1}u_{n+1} + b_nu_n + a_nu_{n-1}, n \geq 0, \tag{3}$$

with real entries, $a_n > 0$, for $n \geq 1$, and $a_0 = 0$.

The role of standard orthogonal polynomials

Inverse problem (Favard's Theorem):
Given a sequence of monic polynomials satisfying a TTRR

$$xP_n(x) = P_{n+1}(x) + b_n P_n(x) + a_n^2 P_{n-1}(x), n \geq 0, \tag{4}$$

with initial conditions $P_{-1}(x) = 0$ and $P_0(x) = 1$, they are orthogonal with respect to a positive Borel measure (not unique in general). What about the spectrum of this measure?
This yields the analysis of the spectrum of continuum analog in one dimension of the discrete Schrödinger operator

$$(Hu)(x) = -u''(x) + V(x)u(x) \tag{5}$$

Dense point spectrum vs purely absolutely continuous spectrum.

The role of standard orthogonal polynomials

Zeros of orthogonal polynomials are real, simple and they are located in the interior of the convex hull of the support E of the measure $d\mu$.
Let us denote $x_{n,1} < x_{n,2} < \cdots < x_{n,n}$ the zeros of $P_n(x)$.

Theorem

The zeros of P_n are the eigenvalues of the leading principal submatrix of size $n \times n$ of the Jacobi matrix.

Theorem

(Interlacing property).
Between two consecutive zeros of P_{n+1} there is exactly one zero of P_n.

Zeros of orthogonal polynomials as nodes in Gaussian quadrature and positivity of the Christoffel-Cotes numbers as a consequence of the interlacing property.

The role of standard orthogonal polynomials

Theorem

Given a decreasing sequence $(s_n)_{n\geq 1}$ and an increasing sequence $(r_n)_{n\geq 1}$ of real numbers, there exists a sequence of monic orthogonal polynomials such that $s_n = x_{n,1}$ and $r_n = x_{n,n}$

On the other hand, it is easy to describe the essential support of $d\mu$ in terms of the zeros of $P_n(x)$ for large n.

Theorem

1. *If $y_0 \in E$, then for every $\varepsilon > 0$ $P_n(x)$ has zeros in $(y_0 - \varepsilon, y_0 + \varepsilon)$ for n large enough.*
2. *If $(a,b) \bigcap E = \emptyset$, then for every n, $P_n(x)$ has at most one zero in (a,b).*
3. *If the sequence $(a_n)_{n\geq 1}$ is bounded and y_0 does not belong to E, then there is a $\delta > 0$, so for each n at most one of P_n and P_{n+1} has a zero in $(y_0 - \delta, y_0 + \delta)$.*

The role of standard orthogonal polynomials

Asymptotics of orthogonal polynomials

1. Outer Strong Asymptotics.

$$\lim_{n\to\infty} \frac{P_n(z)}{\Phi^n(z)}.$$

   Szegő class of measures when E is bounded. Techniques of Classical Complex Analysis.

2. Outer Ratio Asymptotics.

$$\lim_{n\to\infty} \frac{P_{n+1}(z)}{P_n(z)}.$$

   Nevai class of measures when E is bounded. Techniques of difference equations (Poincaré Theorem)

3. N-th root Asymptotics.

$$\lim_{n\to\infty} |P_n(z)|^{1/n}.$$

   Regular class of measures when E is bounded. Techniques of Potential Theory (Green functions)

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Sobolev OP

An inner product is said to be a Sobolev inner product if

$$\langle f, g \rangle_S := \int_{E_0} f(x)g(x)d\mu_0 + \sum_{k=1}^{m} \int_{E_k} f^{(k)}(x)g^{(k)}(x)d\mu_k,$$

where $(d\mu_0, \cdot\cdot\cdot, d\mu_m)$ is a vector of positive Borel measures and E_k= supp $d\mu_k$, $k = 0, 1, \cdot\cdot, m$.

Using Gram-Schmidt method for the canonical basis $(x^n)_{n\geq 0}$ you get a sequence of monic orthogonal polynomials. Thus, the n-th orthogonal polynomials is an extremal polynomial in terms of the Sobolev norm among all monic polynomials of degree exactly n.

Taking into account $\langle xf, g \rangle_S \neq \langle f, xg \rangle_S$ these polynomials do not satisfy a TTRR. Thus, a basic property of standard orthogonal polynomials is lost. A natural question is to compare analytic properties of these polynomials and the standard ones.

A natural framework is the implementation of spectral methods for Boundary Value Problems for elliptic differential operators. Instead of the use of standard orthogonal polynomials when you deal with the Galerkin approximation in the variational problem it seems to be natural to use Sobolev orthogonal polynomials taking into account the matrix problem reduces to a diagonal problem. Unfortunately, there is not a general theory about these families of orthogonal polynomials like in the standard case.

A second interesting problem is related to quadrature formulas involving derivatives. Again, very few results are known about the zeros of Sobolev orthogonal polynomials as nodes in such quadrature rules.

Third, the analysis of Fourier expansions in terms of Sobolev orthogonal polynomials seems to be more accurate than the standard ones in order to increase the speed of convergence as well as to analyze the behavior of the approximation at the end points of the support of the measures, assuming it is a bounded interval.

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The birth time of Sobolev OP

In 1947 D. C. Lewis stated the following problem in the framework of polynomial least square approximations: Let $\alpha_0, \ldots, \alpha_p$ be monotonic, non-decreasing functions defined on $[a, b]$ and let f be a function on $[a, b]$ that satisfies certain regularity conditions. Determine a polynomial P_n of degree $\leq n$ that minimizes

$$\sum_{k=0}^{p} \int_a^b |f^{(k)}(x) - P_n^{(k)}(x)|^2 d\alpha_k(x).$$

Lewis did not use Sobolev orthogonal polynomials and gave a formula for the reminder term of the approximation as an integral of the Peano kernel. The first paper on Sobolev orthogonal polynomials was published by Althammer in 1962, who attributed his motivation to Lewis's paper. These Sobolev OP are orthogonal with respect to the inner product

$$\langle f, g \rangle_S = \int_{-1}^{1} f(x)g(x)dx + \lambda \int_{-1}^{1} f'(x)g'(x)dx, \quad \lambda > 0. \tag{6}$$

The birth time of Sobolev OP

Let $S_n(\cdot;\lambda)$ denote the orthogonal polynomial of degree n with respect to the inner product $\langle\cdot,\cdot\rangle_S$, normalized so that $S_n(1;\lambda) = 1$, and let P_n denote the n-th Legendre polynomial. The following properties hold for $S_n(\cdot;\lambda)$:

1. $\{S_n(\cdot;\lambda)\}_{n\geq 0}$ satisfies a differential equation

$$\lambda S_n''(x;\lambda) - S_n(x;\lambda) = A_n P_{n+1}'(x) + B_n P_{n-1}'(x),$$

   where A_n and B_n are constants that can be given by explicit formulas.

2. $\{S_n(\cdot;\lambda)\}_{n\geq 0}$ satisfies a recursive relation

$$S_n(x;\lambda) - S_{n-2}(x;\lambda) = a_n(P_n(x) - P_{n-2}(x)), \quad n = 1, 2, \ldots.$$

3. $S_n(\cdot;\lambda)$ has n real simple zeros in $(-1,1)$,

The Sobolev-Legendre polynomials were also studied by Gröbner (1967), who established a version of the Rodrigues formula which states that, up to a constant factor c_n,

$$S_n(x;\lambda) = c_n \frac{\partial^n}{1-\lambda\partial^2}\left((x^2-x)^n - \alpha_n(x^2-x)^{n-1}\right)$$

where α_n are real numbers explicitly given in terms of λ and n.

The birth time of Sobolev OP

Althammer also gave an example in which he replaced dx in the second integral in $\langle \cdot, \cdot \rangle_S$ by $w(x)dx$ with $w(x) = 10$ for $-1 \leq x < 0$ and $w(x) = 1$ for $0 \leq x \leq 1$, and made the observation that $S_2(x; \lambda)$ for this new inner product has one real zero outside of $(-1, 1)$.

Brenner considered the inner product

$$\langle f, g \rangle := \int_0^\infty f(x)g(x)e^{-x}dx + \lambda \int_0^\infty f'(x)g'(x)e^{-x}dx, \quad \lambda > 0,$$

and obtained results similar to those of Althammer.

Sobolev orthogonal polynomials: general aspects

Essentially, there are three types of Sobolev inner products (SIP) on the real line:

- Continuous Sobolev inner products. All the measures μ_i with $k \in \{0, 1, \dots, m\}$ have continuous support.
- Sobolev-type inner products. μ_0 has continuous support and μ_i, $i = 1, \dots, m$, are supported on finite subsets.
- Discrete-Continuous Sobolev inner products. μ_m has continuous support and μ_i, $i = 0, \dots, m-1$, are supported on finite subsets.

The methods to deal with these three kind of SIP are quite different.

Sobolev orthogonal polynomials: general aspects

There is an extensive literature about Sobolev orthogonality and the corresponding orthogonal polynomials and their properties. Some general/survey papers

- Alfaro, Marcellán, Rezola, 1993. First survey.
- Meijer 1996. Second survey.
- Martínez–Finkelshtein, 1998, 2001. Surveys on analytical properties of Sobolev orthogonal polynomials (SOP).
- Marcellán, Moreno–Balcázar, 2006. Survey about SOP involving measures with unbounded support.
- Marcellán, Xu, 2015. A general state of knowledge.
- Marcellán, Moreno–Balcázar, 2017.

Algebraic, differential and asymptotic properties of SOP constitute the focus of attention.

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Coherent pairs of measures and Sobolev OP

The study of Sobolev orthogonal polynomials unexpectedly became largely dormant for nearly two decades, from which it reemerged only when a new ingredient, *coherent pairs*, was introduced by Iserles, Koch, Nørsett and Sanz-Serna in 1991 in the framework of Fourier expansions in terms of Legendre-Sobolev orthogonal polynomials.

The comparison with standard Legendre-Fourier expansions reveals the interest of these new expansions despite the fact you have not explicit expressions neither for the polynomials nor the corresponding kernels.

Coherent pairs of measures and Sobolev OP

The coherent pair introduced in 1991 is defined for the inner product

$$\langle f, g \rangle_\lambda = \int_a^b f(x)g(x)d\mu_0(x) + \lambda \int_a^b f'(x)g'(x)d\mu_1(x), \tag{7}$$

where $-\infty \leq a < b \leq \infty$, μ_0 and μ_1 are positive Borel measures on the real line with finite moments of all orders. Let $P_n(\cdot; d\mu_i)$ denote the monic orthogonal polynomial of degree n with respect to $d\mu_i$, i=0,1.

Definition

The pair $\{d\mu_0, d\mu_1\}$ is called coherent if there exists a sequence of nonzero real numbers $\{a_n\}_{n\geq 1}$ such that

$$P_n(\cdot; d\mu_1) = \frac{P'_{n+1}(\cdot; d\mu_0)}{n+1} + a_n \frac{P'_n(\cdot; d\mu_0)}{n}, \qquad n \geq 1. \tag{8}$$

Coherent pairs of measures and Sobolev OP

Definition (cont.)

If $[a,b]=[-c,c]$ and $d\mu_0$ and $d\mu_1$ are both even, then $\{d\mu_0, d\mu_1\}$ is called a symmetrically coherent pair if

$$P_n(\cdot; d\mu_1) = \frac{P'_{n+1}(\cdot; d\mu_0)}{n+1} + a_n \frac{P'_{n-1}(\cdot; d\mu_0)}{n-1}, \quad n \geq 2. \tag{9}$$

In the case of $d\mu_1 = d\mu_0$, we call $d\mu_0$ self-coherent.

Theorem

If $\{d\mu_0, d\mu_1\}$ is a coherent pair, then

$$S_n(x;\lambda) + b_{n-1}(\lambda) S_{n-1}(x;\lambda) = P_n(x; d\mu_0) + \widehat{a}_{n-1} P_{n-1}(x; d\mu_0), \tag{10}$$

where $\widehat{a}_{n-1} = n a_n/(n-1)$ and $b_{n-1}(\lambda) = \widehat{a}_{n-1} \|P_{n-1}(\cdot; d\mu_0)\|^2_{d\mu_0} / \|S_{n-1}(\cdot;\lambda)\|^2_\lambda$.

Coherent pairs of measures and Sobolev OP

Theorem

If $\{\mathcal{U}_0,\mathcal{U}_1\}$ is a coherent pair, then at least one of them has to be classical in the extended sense.

When $\mathcal{U}_0$ and $\mathcal{U}_1$ are positive definite linear functionals associated with measures $d\mu_0$ and $d\mu_1$, these cases are given as follows (Meijer 1997):

Laguerre case

Type I

1. $d\mu_0(x) = (x-\xi)x^{\alpha-1}e^{-x}dx$ and $d\mu_1(x) = x^{\alpha}e^{-x}dx$, where if $\xi < 0$ then $\alpha > 0$, and if $\xi = 0$ then $\alpha > -1$.
2. $d\mu_0(x) = e^{-x}dx + M\delta_0$ and $d\mu_1(x) = e^{-x}dx$, where $M \geq 0$.

Type II

1. $d\mu_0(x) = x^{\alpha}e^{-x}dx$ and $d\mu_1(x) = \frac{x^{\alpha+1}e^{-x}}{x-\xi}dx + M\delta_{\xi}$, where if $\xi < 0$, $\alpha > -1$ and $M \geq 0$.

Coherent pairs of measures and Sobolev OP

Jacobi case

Type I

1. $d\mu_0(x) = |x-\xi|(1-x)^{\alpha-1}(1+x)^{\beta-1}dx$ and $d\mu_1(x) = (1-x)^{\alpha}(1+x)^{\beta}dx$, where if $|\xi| > 1$ then $\alpha > 0$ and $\beta > 0$, if $\xi = 1$ then $\alpha > -1$ and $\beta > 0$, and if $\xi = -1$ then $\alpha > 0$ and $\beta > -1$.
2. $d\mu_0(x) = (1+x)^{\beta-1}dx + M\delta_1$ and $d\mu_1(x) = (1+x)^{\beta}dx$, where $\beta > 0$ and $M \geq 0$.
3. $d\mu_0(x) = (1-x)^{\alpha-1}dx + M\delta_{-1}$ and $d\mu_1(x) = (1-x)^{\alpha}dx$, where $\alpha > 0$ and $M \geq 0$.

Type II

1. $d\mu_0(x) = (1-x)^{\alpha}(1+x)^{\beta}dx$ and $d\mu_1(x) = \frac{1}{|x-\xi|}(1-x)^{\alpha+1}(1+x)^{\beta+1}dx + M\delta_{\xi}$, where $|\xi| > 1$, $\alpha > -1$ and $\beta > -1$, and $M \geq 0$.

Symmetrically coherent pairs

The similar analysis was also carried out for symmetrically coherent pairs. It leads to the following list of symmetrically coherent pairs.

Hermite case

1. $d\mu_0(x) = e^{-x^2}dx$ and $d\mu_1(x) = \frac{1}{x^2+\xi^2}e^{-x^2}dx$, where $\xi \neq 0$.
2. $d\mu_0(x) = (x^2+\xi^2)e^{-x^2}dx$ and $d\mu_1(x) = e^{-x^2}dx$, where $\xi \neq 0$.

Symmetrically coherent pairs

Gegenbauer case

1. $d\mu_0(x) = (1-x^2)^{\alpha-1}dx$ and $d\mu_1(x) = \frac{1}{x^2+\xi^2}(1-x^2)^{\alpha}dx$, where $\xi \neq 0$ and $\alpha > 0$.
2. $d\mu_0(x) = (1-x^2)^{\alpha-1}dx$ and $d\mu_1(x) = \frac{1}{\xi^2-x^2}(1-x^2)^{\alpha}dx + M\delta_{\xi} + M\delta_{-\xi}$, where $|\xi| \geq 1$, $\alpha > 0$ and $M \geq 0$.
3. $d\mu_0(x) = (x^2+\xi^2)(1-x^2)^{\alpha-1}dx$ and $d\mu_1(x) = (1-x^2)^{\alpha}dx$, where $\alpha > 0$.
4. $d\mu_0(x) = (\xi^2-x^2)(1-x^2)^{\alpha-1}dx$ and $d\mu_1(x) = (1-x^2)^{\alpha}dx$, where $|\xi| \geq 1$ and $\alpha > 0$.
5. $d\mu_0(x) = dx + M\delta_1 + M\delta_{-1}$ and $d\mu_1(x) = dx$, where $M \geq 0$.

Fourier series and coherent pairs

For a coherent pair (μ_0, μ_1) of measures of Jacobi type, let introduce the space $W^p_{0,1} = \{f : [-1, 1] \to \mathbb{R}, ||f||^p_{0,1} = ||f||^p_{\mu_0} + \lambda||f'||^p_{\mu_1} < \infty\}$.

Let consider the nth Fourier partial sum $S_n f$ of a function $f \in W^p_{0,1}$ in terms of the sequence of Sobolev orthonormal polynomials associated with the Jacobi coherent pair.

We are interested to analyze the convergence of the Fourier partial sums with the above Sobolev norm.

Fourier series and coherent pairs

Theorem

(Ciaurri, Mínguez 2021)
Let $\alpha > -1, \beta > 0$ *and* $1 < p < \infty$.

- *If* (μ_0, μ_1) *is a Jacobi coherent pair of type I, then* $||S_n f||_{W^p_{0,1}} \leq C||f||_{W^p_{0,1}}$ *with a constant* C *independent of* n *and* f *if and only if the following conditions hold*

$$|(\alpha+1)(\frac{1}{p}-\frac{1}{2})| < \frac{1}{4}, |(\beta+1)(\frac{1}{p}-\frac{1}{2})| < \frac{1}{4}. \tag{11}$$

- *If* (μ_0, μ_1) *is a Jacobi coherent pair of type II, with* $M = 0$, *then* $||S_n f||_{W^p_{0,1}} \leq C||f||_{W^p_{0,1}}$ *with a constant* C *independent of* n *and* f *if and only if the following conditions hold*

$$|(\alpha+2)(\frac{1}{p}-\frac{1}{2})| < \frac{1}{4}, |(\beta+1)(\frac{1}{p}-\frac{1}{2})| < \frac{1}{4}. \tag{12}$$

If the Sobolev space is complete an the polynomials constitute a dense subset then the uniform boundedness of S_n would be equivalent to the convergence in $W^p_{0,1}$. Indeed,

Theorem

(Ciaurri, Mínguez 2021)
Let $\alpha > -1, \beta > 0$ and $1 < p < \infty$ and (μ_0, μ_1) be a coherent pair of Jacobi measures where $M = 0$ for type II. Then

- *The set of polynomials is dense in the space $W^p_{0,1}$.*
- *$W^p_{0,1}$ is a complete space.*

Fourier series and coherent pairs

Theorem

(Ciaurri, Mínguez 2021)
Let $\alpha > -1, \beta > 0$ *and* $1 < p < \infty$ *and* (μ_0, μ_1) *be a coherent pair of Jacobi measures where* $M = 0$ *for type II. Then*

$$\lim_{n\to\infty} ||S_n f - f||_{W^p_{0,1}} = 0$$

if and only if

- *For type I,* (11) *holds*
- *For type II,* (12) *holds*

Open problems

- What happens when you have $M \neq 0$ for type II Jacobi coherent pair of measures?
- To analyze the convergence of the Fourier series expansion in the Sobolev space for Laguerre coherent pairs of measures.
- To analyze the convergence of the Fourier series expansion in the Sobolev space for Hermite coherent pairs of measures.

Generalized coherent pairs

We can restate (10) as

$$S_n(x;\lambda)+b_{n-1}(\lambda)S_{n-1}(x;\lambda)=P_n(x;d\mu_0)+\widehat{a}_{n-1}P_{n-1}(x;d\mu_0),\quad n\geq 1. \quad (10')$$

Let $\mathsf{S}_n(x)$ denote the left hand side of (10'). Clearly S_n' can be expanded in terms of $\{P_k(\cdot;d\mu_1)\}$,

$$\mathsf{S}_n'(x)=nP_{n-1}(x;d\mu_1)+\sum_{k=0}^{n-2}d_{k,n}P_k(x;d\mu_1),\qquad d_{k,n}=\frac{\langle \mathsf{S}_n',P_k(\cdot;d\mu_1)\rangle_{d\mu_1}}{\|P_k(\cdot;d\mu_1)\|^2_{d\mu_1}}.$$

We can conclude the following relation between $\{P_n(\cdot;d\mu_0)\}$ and $\{P_n(\cdot;d\mu_1)\}$

$$P_n(x;d\mu_1)+b_{n-1}P_{n-1}(x;d\mu_1)=\frac{P_{n+1}'(x;d\mu_0)}{n+1}+a_n\frac{P_n'(x;d\mu_0)}{n},\quad n\geq 1. \quad (13)$$

Generalized coherent pairs

Definition

The pair $\{d\mu_0, d\mu_1\}$ is called a generalized coherent pair if (13) holds for all $n \geq 1$, and this definition extends to the linear functionals $\{\mathcal{U}_0, \mathcal{U}_1\}$.

Theorem

If $\{\mathcal{U}_0, \mathcal{U}_1\}$ is a generalized coherent pair, then at least one of them has to be semiclassical of class at most 1 and the other one is a rational perturbation of it. (Delgado, Marcellán, 2004)

All generalized coherent pairs of linear functionals are listed in such a paper.

Some examples of symmetrically generalized coherent pairs have been studied by E. X. L. Andrade , C. F. Bracciali and A. Sri Ranga (2009) when $\mathcal{U}_0$ is associated with the Gegenbauer weight and $\mathcal{U}_1 = \frac{1-x^2}{1+qx^2}\mathcal{U}_0$ + two Dirac deltas with the same masses and supported at the zeros of $1 + qx^2$ if $-1 \leq q < 0$. If $q > 0$ you have not mass points.

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Sobolev type OP

An inner product is called a Sobolev type inner product if the derivatives appear only on function evaluations on a finite discrete set. More precisely, such an inner product takes the form

$$\langle f, g \rangle_S := \int_{\mathbb{R}} f(x)g(x)d\mu_0 + \sum_{k=1}^{m} \int_{\mathbb{R}} f^{(k)}(x)g^{(k)}(x)d\mu_k, \tag{14}$$

where $d\mu_0$ is a positive Boreal measure supported on an infinite subset of the real line and $d\mu_k$, $k = 1, 2, ..., m$, are positive Borel measures supported on finite subsets of the real line.

Sobolev type OP

The first study was carried out for the classical weight functions. The Laguerre case was studied by Koekoek (1990) with $d\mu_0 = x^\alpha e^{-x} dx$, $\alpha > -1$, and $d\mu_k = M_k \delta_0, k = 1, 2, ..., m$; the n-th Sobolev orthogonal polynomial, S_n, is given by

$$S_n(x) = \sum_{k=0}^{\min\{n,m+1\}} (-1)^k A_{n,k} L_{n-k}^{(\alpha+k)}(x),$$

in which A_k are constants determined by a linear system of equations. The Gegenbauer case was studied by Bavinck and Meijer (1989/90) with $d\mu_0 = (1-x^2)^{\lambda-1/2} dx + A(\delta_{-1} + \delta_1)$, $\lambda > -1/2$, and $m = 1$, $d\mu_1 = B(\delta_{-1} + \delta_1)$; the n-th Sobolev orthogonal polynomial is given by

$$S_n(x) = \sum_{k=0}^{2} a_{k,n} x^k C_{n-k}^{(\lambda+k)}(x)$$

where $a_{0,n}$, $a_{1,n}$, $a_{2,n}$ are appropriate constants. In both cases, the Sobolev orthogonal polynomials satisfy higher (than three) order recurrence relations that expands $q(x)S_n(x)$ as a sum of S_m.

When $M_k = 0$, $k = 1, 2, ..., m-1$, and $d\mu_m = M_m \delta_c$, the inner product (14) becomes

$$\langle f, g \rangle_m := \int_{\mathbb{R}} f(x)g(x)d\mu_0 + M_m f^{(m)}(c)g^{(m)}(c), \tag{15}$$

where $c \in \mathbb{R}$ and $M_m \geq 0$.

For $i, j \in \mathbb{N}_0$, define

$$K_{n-1}^{(i,j)}(x,y) := \sum_{l=0}^{n-1} \frac{P_l^{(i)}(x) P_l^{(j)}(y)}{||P_l||_{d\mu_0}^2}.$$

Sobolev type OP

It was shown by Marcellán and Ronveaux (1990) that

$$S_n(x) = P_n(x) - \frac{M_m P_n^{(m)}(c)}{1 + M_m K_{n-1}^{(m,m)}(c,c)} K_{n-1}^{(0,m)}(x,c), \tag{16}$$

which extends the expression for $m = 0$ by Krall (1980/81). From this relation, one deduces immediately that

$$S_{n+1}(x) + a_n S_n(x) = P_{n+1}(x) + b_n P_n(x), \qquad n \geq 0,$$

where a_n and b_n are constants that can be easily determined.
The Sobolev polynomials S_n also satisfy a higher order recurrence relation

$$(x-c)^{m+1} S_n(x) = \sum_{j=n-m-1}^{n+m+1} c_{n,j} S_j(x), \tag{17}$$

where $c_{n,n+m+1} = 1$ and $c_{n,n-m-1} \neq 0$.

Sobolev type OP

There are two types of results in this direction, both related to Sobolev orthogonal polynomials.

The first one gives a characterization of an inner product $\langle \cdot, \cdot \rangle$ for which orthogonal polynomials satisfy the recurrence relation of the form (17), which holds if the operator of multiplication by $M_{m,c} := (\cdot - c)^{m+1}$ is symmetric, *i.e.*, $\langle M_{m,c}p, q \rangle = \langle p, M_{m,c}q \rangle$. It was proved by Durán (1993) that if $\langle \cdot, \cdot \rangle$ is an inner product such that $M_{m,c}$ is symmetric and it commutes with the operator $M_{0,c}$, *i.e.*, $\langle M_{m,c}p, M_{0,c}q \rangle = \langle M_{0,c}p, M_{m,c}q \rangle$, then there exists a nontrivial positive Borel measure $d\mu_0$ and a real, positive semi-definite matrix A of size $m+1$, such that the inner product is of the form

$$\langle p, q \rangle = \int_{\mathbb{R}} p(x)q(x)d\mu_0 + \left(p(c), p'(c), \ldots, p^{(m)}(c)\right) A \left(q(c), q'(c), \ldots, q^{(m)}(c)\right)^T .$$

Sobolev type OP

Furthermore, a connection between such Sobolev orthogonal polynomials and matrix orthogonal polynomials was established by Durán and Van Assche (1995), by representing the higher order recurrence relation as a three term recurrence relation with matrix coefficients for a family of matrix orthogonal polynomials defined in terms of the Sobolev orthogonal polynomials.

The second type of Favard type theorem was given by Evans, Littlejohn, Marcellán, Markett and Ronveaux (1995), where it was proved that the operator of multiplication by a polynomial h is symmetric with respect to the inner product (14) if and only if $d\mu_k, k = 1, 2, .., m$, are discrete measures whose supports are related to the zeros of h and their derivatives. Consequently, higher order recurrence relations for Sobolev inner products appear only in Sobolev inner product of the second type.

Contents

Sobolev type inner products

The first work on asymptotics for Sobolev orthogonal polynomials was carried out by Marcellán and Van Assche (1993) for the inner product

$$\langle f, g\rangle_S = \int_{-1}^{1} f(x)g(x)d\mu_0(x) + M_1 f'(c)g'(c),$$

where $c \in \mathbb{R}$, $M_1 > 0$ and the measure $d\mu_0$ belongs to the Nevai class $M(0,1)$. If $c \in \mathbb{R} \setminus \operatorname{supp} \mu_0$, then

$$\lim_{n\to\infty} \frac{S_n(z)}{P_n(z, d\mu_0)} = \frac{(\Phi(z) - \Phi(c))^2}{2\Phi(z)(z-c)}, \qquad \Phi(z) := z + \sqrt{z^2-1},$$

locally uniformly outside the support of the measure, where $\sqrt{z^2-1} > 0$ when $z > 1$.

If $c \in \operatorname{supp}\mu_0$, then $\lim_{n\to\infty} \frac{S_n(z)}{P_n(z; d\mu_0)} = 1$ outside the support of the measure.

Some Extensions

- **Bounded support.**

1. $A \in \mathbb{R}^{(N,N)}$

$$\lim_{n\to\infty} \frac{S_n(z)}{P_n(z, d\mu_0)} = \left(\frac{(\Phi(z) - \Phi(c))^2}{2\Phi(z)(z-c)} \right)^r, \qquad r := \text{rankA},$$

locally uniformly outside the support of the measure.

2. López Lagomasino, Marcellán and Van Assche (1995)

$$\langle f, g \rangle = \int f(x)g(x)d\mu_0(x) + \sum_{j=1}^{N} \sum_{k=0}^{N_j} f^{(k)}(c_j) L_{j,k}(g; c_j),$$

where $d\mu_0 \in M(0,1)$, $\{c_k\}_{k=1}^N \in \mathbb{R} \setminus \operatorname{supp} \mu_0$, $j = 1, \ldots, N$, and $L_{j,k}$ is an ordinary differential operator. Then

$$\lim_{n\to\infty} \frac{S_n^{(\nu)}(z)}{P_n^{(\nu)}(z, d\mu_0)} = \prod_{j=1}^{m} \left(\frac{(\Phi(z) - \Phi(c))^2}{2\Phi(z)(z-c)} \right)^{I_j},$$

Some Extensions

where I_j is the dimension of the square matrix obtained from the matrix of the coefficients of L_{j,N_j} after deleting all zero rows and columns.

On the other hand if both the measure $d\mu_0$ and its support Δ are regular, then

$$\limsup_{n\to\infty} \|S_n^{(j)}\|_{\Delta}^{1/n} = C(\Delta), \qquad j \geq 0,$$

where $\|\cdot\|_{\Delta}$ denotes the uniform norm in the support of the measure and $C(\Delta)$ is its logarithmic capacity (López-Lagomasino, Pijeira, 1999).

Some Extensions

- **Unbounded support** (Marcellán, Moreno-Balcázar, 2006).

$$d\mu_0 = x^\alpha e^{-x}dx, \quad \alpha > 1, \quad c = 0, \quad A \in \mathbb{R}^{(2,2)}.$$

1. Outer relative asymptotics.
2. Outer relative asymptotics for scaled polynomials.
3. Mehler-Heine formula.
4. Inner strong asymptotics.

If $c < 0$, then

$$\lim_{n\to\infty} \frac{S_n(z)}{L_n^{(\alpha)}(z)} = \left(\frac{\sqrt{-z}-\sqrt{-c}}{\sqrt{-z}+\sqrt{-c}}\right)^r, \qquad r = \operatorname{rank} A,$$

locally uniformly on compact subsets of the exterior of $\mathbb{R}_+$ (Marcellán, Zejnullahu, Fejzullahu, Huertas, 2012).

Continuous Sobolev inner products

- Bounded support - coherent case (Martínez-Finkelshtein, Moreno-Balcázar, Pérez, Piñar, 1998).

$$\lim_{n\to\infty} \frac{S_n(z)}{P_n(z;d\mu_1)} = \frac{2}{\Phi'(z)},$$

  locally uniformly outside $[-1,1]$.

- If the measures μ_0 and μ_1 are absolutely continuous and belong to the Szegő class, the above result is also true (Martínez-Finkelshtein, 2000). The role is played by the measure $d\mu_1$!!

- Unbounded support - coherent case.
  1. Outer relative asymptotics.
  2. Scaled outer asymptotics.
  3. Inner strong asymptotics.

Contents

The Cauchy problem and SOP

Sharapudinov (2017, 2018) considers the Cauchy problem or initial value problem (IVP)

$$\begin{cases} F(x, y, y', \dots, y^{(r)}) = 0, \\ y^{(m)}(-1) = y_m, \quad m = 0, \dots, r-1. \end{cases}$$

He establishes that the Fourier series

$$y(x) = \sum_{k=0}^{\infty} \hat{y_k} T_{r,k}(x),$$

where $\{T_{r,n}\}_n$ is the sequence of SOP with respect to the discrete–continuous SIP

$$(f, g)_S = \sum_{i=0}^{r-1} f^{(i)}(-1) g^{(i)}(-1) + \int_{-1}^{1} f^{(r)}(x) g^{(r)}(x) \frac{2}{\pi\sqrt{1-x^2}} dx,$$

is an efficient tool to give an approximation to the solution of IVP. The same author has considered IVP for other weights (Laguerre) as well as for discrete weights (Meixner).

BVP for second order elliptic problems and SOP

The Chinese mathematicians X. Yu, Z. Wang and H. LI (2019) used basis of SOP to analyze spectral methods for second-order elliptic BVP problems in the interval $I = [-1, 1]$. For example, they deal with a second order problem with a harmonic potential,

$$\begin{cases} -u''(x) + \mu x^2 u(x) = f(x), & \mu \geq 0, x \in I, \\ u(-1) = u(1) = 0. \end{cases}$$

Let consider the space $H_0^1(I) = \{g : I \to \mathbb{R}, g, g' \in L^2(I), g(\pm 1) = 0.$

The basic idea is to have an weak (variational) formulation of the above problem, i.e. to find a function $u \in H_0^1$ such that $B(u, v) = (u', v') + \mu(x^2 u, v) = (f, v)$ for every $v \in H_0^1$.

The generalized Jacobi spectral method for the above formulation yields to find a polynomial u_N of degree N, with $u_N(\pm 1) = 0$, such that $B(u_N, q) = (f, q)$, for every polynomial of degree N satisfying $q(\pm 1) = 0$.

Notice that the above polynomials belong to the span $\{P_n^{(-1,-1)}(x), 2 \leq n \leq N\}$.

Theorem

The sequence of polynomials $\{P_n^{(-1,-1)}(x), 2 \leq n \leq N\}$ *is orthogonal with respect to the discrete-continuous Sobolev inner product*

$$A(p,q) = ap(1)q(1) + bp(-1)q(-1) + \int_{-1}^{1} p'(x)q'(x)dx,$$

assuming $a + b > 0, a + b + 2ab > 0,$

Let us remind that for $n \geq 2$ $P_n^{(-1,1)}(x)$is, up to a constant factor, $(x^2 - 1)P_{n-2}^{(1,1)}(x)$.

BVP for second order elliptic problems and SOP

Let consider the set of polynomials $\{\Phi_n\}$ such that $B(\Phi_n, \Phi_m) = \rho_n \delta_{m.n}, m, n \geq 2$, with $\Phi_2(x) = (3)_2 P_2^{(-1,-1)}(x)$, and $\Phi_n(x) - (n+1)_n P_n^{(-1,-1)}(x), n \geq 3$, a polynomial of degree $n-1$ vanishing at ± 1.

Theorem

$(n+1)_n = P_n^{(-1,-1)}(x) = \Phi_n(x) + d_{n,1}\Phi_{n-2}(x) + d_{n,2}\Phi_{n-4}(x), n \geq 2.$

As a consequence

Theorem

If u and u_N are the solutions of the above two variational problems, then

$$u_N(x) = \sum_{n=2}^{N} \frac{(f, \Phi_n)}{\rho_n} \Phi_n(x) \quad \textit{and} \quad u(x) = \sum_{n=2}^{\infty} \frac{(f, \Phi_n)}{\rho_n} \Phi_n(x).$$

This constitutes an alternative approach to the Galerkin or variational methods based on standard orthogonal polynomials. The computational cost is related to the construction of the Sobolev orthogonal polynomials.

Discrete-continuous SOP and Fourier series

In A. Díaz-González, F. Marcellán, H. Pijeira-Cabrera, W. Urbina, *Discrete-Continuous Jacobi-Sobolev Spaces and Fourier Series*, Bull. Malaysian Math. Sci. Soc. **44** (2) (2021), 571-598, the following discrete-continuous Sobolev inner product is considered.

$$\langle f, g\rangle = \sum_{k=0}^{N-1} f^{(k)}(\omega_k)\, g^{(k)}(\omega_k) + \int_{-1}^{1} f^{(N)}(x) g^{(N)}(x)(1-x)^{\alpha}(1+x)^{\beta} dx.$$

The completeness of the Sobolev space associated with the above discrete-continuous Sobolev norm, the denseness of the polynomials as well as the conditions for the convergence in such a norm of the nth partial Fourier-Sobolev sum in terms of the family of orthogonal polynomials with respect to $\langle , \rangle$ have been analyzed.

Some useful references

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Thanks