Seminario Iberoamericano de
Análisis Matemático y Matemática Aplicada

Fourier Series and Orthogonal Polynomials

A long standing and fuitful history

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1.-Convergence of Fourier Series

- “The crisis struck four days before Christmas 1807. The edifice of calculus was shaken to its foundations. In retrospect, the difficulties had been building for decades. Yet while most scientists realized that something had happened, it would take fifty years before the full impact of the event was understood. The nineteenth century would see ever expanding investigations into the assumptions of calculus, an inspection and refitting of the structure from the footings to the pinnacle, so thorough a reconstruction that calculus was given a new name: Analysis. Few of those who witnessed the incident of 1807 would have recognized mathematics as it stood one hundred years later. The twentieth century was to open with a redefinition of the integral by Henri Lebesgue and an examination of the logical underpinnings of arithmetic by Bertrand Russell and Alfred North Whitehead, both direct consequences of the events set in motion in that critical year. The crisis was precipitated by the deposition at the Institut de France in Paris of a manuscript, Theory of the Propagation of Heat in Solid Bodies, by the 39-year old prefect of the department of IsÃ¨re, Joseph Fourier.”

  A Radical Approach to Real Analysis
  David M. Bressoud

- “It was this pliability which was embodied in Fourier’s intuition, commonly but falsely called a theorem, according to which the trigonometric series çan express any function whatever between definite values of the variable."This familiar statement of Fourier’s "theorem,"taken from Thompson and Tait’s "Natural Philosophy,ïs much too broad a one, but even with the limitations which must to-day be imposed upon the conclusion, its importance can still be most fittingly described as follows in their own words: The theorem ïs not only one of the most beautiful results of modern analysis, but may be said to furnish an indispensable instrument in the treatment of nearly recondite question [sic] in modern physics. To mention only sonorous vibrations, the propagation of electric signals along a telegraph wire, and the conduction of heat by the earth’s crust, as subjects in their generality intractable without it, is to give but a feeble idea of its importance."

  Address to the American Association for the Advancement of Science, 1913
  Edward B. Van Vleck

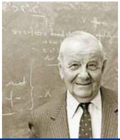
Introduction

- The problem of convergence of Fourier series was probably one of the most important problem no only in analysis but in mathematics.
- Jean-Batiste Joseph Fourier submitted his famous paper *Mémoire sur la propagation de la chaleur dans corps solid (Trearise on the propagation of heat in solid bodies)* in 1807 to the *Institut de France* and in the spring of 1808, Simeon Denis Poisson wrote up the committee's report on it, the conclusion was that it contained nothing that was new or interesting and was rejected. Behind this opinion lay Lagrange's opposition to the admission of Fourier's trigonometric series and his conviction that they must not converge.
- The problem of modeling the flow of heat was of concern to many scientists of the time. In 1811, the Institut de France announced a competition for the best explanation of heat diffusion. Fourier reworked his earlier manuscript and submitted it. Despite continuing objections from Lagrange, he was awarded the prize. Lagrange could not deny him the award, but he could postpone publication. Even after Lagrange died in 1813, Fourier's manuscript continued to languish in the Institut.
- In 1822 Fourier was elected perpetual secretary of the *AcadÃ©mie des Sciences,* the highest of scientific honors and then he was able to publish his book *ThÃ©orie analytique de la chaleur (Analytic theory of heat)*. He used that position in succeeding years to encourage and promote the careers of emerging mathematicians. Gustav Dirichlet, Sophie Germain, Joseph Liouville, Claude Navier, and Charles Sturm were among those who received his assistance and would remember him fondly.
- The problem of the convergence of Fourier series was given its first published treatment in 1820 in a paper by Poisson. His work suffers from the defect that in the course of proving the convergence of Fourier series he needed-in a subtle way-to assume that they converged. Fourier tried to supply a proof in his book. He did see the fundamental difficulty and so was able to show the way to an eventual proof, but he himself did succeed. In 1826, Cauchy took up this problem and published what he considered a solution but there were flaws in his work.

- In January of 1829, at the age of 23 and from his new professorship in Berlin, Gustav Lejeune Dirichlet submitted the paper *Sur la convergence des series trigonometriques qui servent it representer une fonction arbitraire entre des limites donnees* It begins with a critique of flaws in Cauchy's paper, after that Dirichlet goes on to give the first correct proof for the validity of Fourier series. He proved that as long as a function is piecewise monotonic in a closed and bounded interval, the Fourier series converges to the original function. Dirichlet belived that functions did not have to be piecewise monotonic to converge to the original function, but neither he nor anyone else had been to weaken this assumption.
- For Dirichlet, this paper was more than an answer to an abstract question in mathematics; it was a tribute to a mentor and friend, a validation of the new and disturbing series that Joseph Fourier had introduced to the scientific community in 1807.
- In the early 1850's, Bernard Riemann, a young protegé of Dirichlet and a student of K. F. Gauss, would make substantial progress in extending our understanding of trigonometric series. In so doing, the certainties of calculus into question. Over the next 60 years, several questions would emerge and being answered. The answers would be totally unexpected. This is the beginning of real analysis.
- Among those questions the nature of the integration was crucial. Toward the end of the nineteenth century it became clear to many mathematicians that the Riemann integral (about which one learns in calculus courses) should be replaced by some other type of integral, more general and more flexible, better suited for dealing with limit processes. Among the attempts made in this direction, it was Lebesgue's construction which turned out to be the most successful. Further developments of the theory were due to D. Hilbert, L. Fejér, M. Riesz , A. Zygmund, A. P. Calderón among several others.





Definition 1

Let us consider the trigonometric system, that is to say, the system of functions

$$e^{ikx} = \cos kx + i \sin kx \quad (k = 0, \pm 1, \pm 2, \dots).$$

These functions are all periodic, with period 2π, and it is immediate that they form an *orthogonal system* over the interval $Q = [-\pi, \pi]$ of length 2π, since if k and m are distinct integers, then

$$\int_{-\pi}^{\pi} e^{ikx} \overline{e^{imx}} dx = \left[\frac{e^{i(k-m)x}}{k-m} \right]_a^{a+2\pi} = 0$$

We will call it the *complex form* of the trigonometric system.
The system is not orthonormal since $\lambda_k = \int_{-\pi}^{\pi} \left| e^{ikx} \right|^2 dx = 2\pi$ for all k, but then if $\phi_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}$, $\{\phi_k\}$ is an orthonormal system.

Definition 2

The Fourier coefficients of a function $f \in L^1[-\pi, \pi]$ are defined as

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{e^{inx}} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx, \quad \text{for all } n \in \mathbb{Z}$$

and the Fourier series of f is given by

$$f(x) \sim \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx}.$$

As usual, in order to study the convergence of the series we need to consider the partial sum of the series, and find a good representation of it in order to be able to handle it.

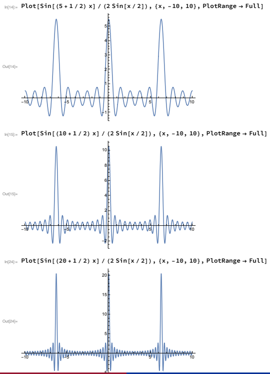
The n-th partial sum of f defined by

$$s_n(f,x) = \sum_{k=-n}^{n} \hat{f}(k)e^{ikx} = \sum_{k=-n}^{n} \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikt}dt\right] e^{ikx} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)[\sum_{k=-n}^{n} \frac{e^{ik(x-t)}}{2}]dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)D_n(x-t)dt$$

where

$$D_n(t) = \sum_{k=-n}^{n} \frac{e^{ik(x-t)}}{2} = \frac{1}{2} + Re\left(\sum_{k=1}^{n} e^{ikt}\right) = \frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin t/2},$$

is the *Dirichlet kernel,*



- In 1876 Dubois Raymond obtained a result that was a breaking point in the development of the Fourier analysis. There exists a continuous periodic f such that its Fourier series diverges (more specifically, the partial sums of $S[f]$ are unbounded) at some point on $[-\pi,\pi]$. After that the research in Fourier analysis stoped completely for more that 30 years, it only resumed after the Lebesgue's measure theory appeared.
- Therefore, another "notions" of convergence were needed. The first one was in "square mean" i.e.

$$\|s_n(f)-f\|_2 = \left(\int_{-\pi}^{\pi} |s_n(f,x)-f(x)|^2\,dx\right)^{1/2} \to 0, \text{ as } n\to\infty.$$

and then, the theory of Hilbert spaces was developed. g.

- Despite the incompleteness of $C[-\pi,\pi]$ Hilbert himself did always work with continuous functions, but since him mathematicians have followed F. Riesz in recognizing that the right setting for this inner product is the larger space $L^2[-\pi,\pi]$, which contains many discontinuous functions but is complete. Nevertheless, earliest attempts along these lines were unsuccessful. The set of Riemann integrable functions f on (0,1) such that

$$\int_{-\pi}^{\pi} |f(t)|^2\,dt < \infty,$$

though much larger than $C[-\pi,\pi]$, is still incomplete with respect to the inner product

$$\langle f,g\rangle = \int_{-\pi}^{\pi} f(t)\overline{g(t)}\,dt.$$

- This was one of the reasons that analysts became dissatisfied with Riemann's definition of integral in the late nineteenth century. Many people worked at trying to develop a better integral, and the race was won by H. Lebesgue in 1902. His integral is much more complicated to define and develop than Riemann's, but as a tool it is easier to use as it has better properties, and in particular allows us to define a space of functions on which $\langle\cdot,\cdot\rangle$ is an inner product determining a complete metric.

- Remember, since the trigonometric system is orthogonal then it satisfies Bessel inequality

$$\left(\sum_{k=1}^{n} \left|\hat{f}(k)\right|^2\right)^{1/2} \leq \|f\|_2$$

for any $n \in \mathbb{N}$ and therefore

$$\left(\sum_{k=1}^{\infty} \left|\hat{f}(k)\right|^2\right)^{1/2} \leq \|f\|_2$$

- Moreover, it can be proved that the trigonometric system is *complete* (i.e. for $f \in L^2[-\pi,\pi]$, such that $\langle f, \phi \rangle = 0 \Rightarrow f = 0$) which is equivalent to *Parseval identity*

$$\left(\sum_{k=1}^{\infty} \left|\hat{f}(k)\right|^2\right)^{1/2} = \|f\|_2$$

holds for any $f \in L^2[-\pi,\pi]$ and that is equivalent to

$$\|s_n(f) - f\| = \left(\int_{-\pi}^{\pi} |s_n(f,x) - f(x)|^2 \, dx\right)^{1/2} \to 0 \text{ as } n \to \infty$$

for any $f \in L^2[-\pi,\pi]$.

- Thus, the basis of the proof of convergence of the partial sums $s_n(f)$ to f in the L^2-norm is the orthogonality of the trigonometric system in $L^2[-\pi,\pi]$.

- The extension of this problem to convergence of the partial sums $s_n(f)$ to f in the L^p-norm $p \neq 2$

$$\|s_n(f) - f\|_p = \left(\int_{-\pi}^{\pi} |s_n(f,x) - f(x)|^p \, dx \right)^{1/p} \to 0, \text{ as } n \to \infty,$$

is a much harder problem since we are now in a Banach space and then we lose the geometry of the Hilbert space!!
- The solution of this problem needs to use crucial notions of harmonic analysis like maximal functions and singular integrals.
- Using the Banach-Steinhaus uniform boundedness principle the L^p convergence of the partial sums is equivalent to prove they are L^p bounded, i.e.

$$\|s_n(f)\|_p \leq c\|f\|_p.$$

- In order to prove that, given that the Dirichlet kernel is bad, it is better to consider

$$s_n^{\#}(f,x) = \frac{s_n(f,x) + s_{n-1}(f,x)}{2} = s_n(f,x) - (\hat{f}(-n)e^{-inx} + \hat{f}(n)e^{inx})$$

- We have

$$\begin{aligned} s_n^{\#}(f,x) &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x-t) \frac{D_n(t) + D_{n-1}(t)}{2} dt \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x-t) \frac{\sin nt}{2\tan \frac{1}{2} t} dt = \frac{\sin nx}{\pi} \int_{-\pi}^{\pi} \frac{f(t)\cos nt}{2\tan \frac{1}{2}(x-t)} dt \\ &\qquad - \frac{\cos nx}{\pi} \int_{-\pi}^{\pi} \frac{f(t)\sin nt}{2\tan \frac{1}{2}(x-t)} dt \end{aligned}$$

- In this context is that the notion of the *conjugated function* appears

$$\tilde{f}(x) = p.v.\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{f(x-t)}{2\tan\frac{t}{2}}dt = \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{\pi>|t|>\varepsilon}\frac{f(x-t)}{2\tan\frac{t}{2}}dt.$$

and its boundedness on $L^p[-\pi.\pi]$ is needed. Assuming this then we have, $\left\|s_n^{\#}(f)\right\|_p \leq C\|f\|_p$ and since $\left\|s_n^{\#}(f) - s_n(f)\right\|_p \leq c\|f\|_1 \leq C\|f\|_p$ we get $\|s_n(f)\|_p \leq C_p\|f\|_p$.

Theorem 1 (M. Riesz)

Given $1 < p < \infty$ and $f \in L^p[-\pi,\pi]$ then the Fourier series of f converges in L^p-norm i.e. $\|s_n(f) - f\|_p \to 0$ as $n \to \infty$.

- The original proof of this result by M. Riesz in 1927 used complex methods, later was obtained using real methods developed by A. P. Calderón and A. Zygmund.
- Since near zero $\frac{1}{2\tan\frac{x}{2}} \sim \frac{1}{x}$ this notion can be extended to the non-periodic case with the definition of the *Hilbert transform,*

$$Hf(x) = p.v.\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(x-y)}{y}dy = \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|y|>\varepsilon}\frac{f(x-y)}{y}dy = \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|x-y|>\varepsilon}\frac{f(y)}{x-y}dy.$$

- From there we have the *Riesz transforms* in $\mathbb{R}^d$

$$R_jf(x) = p.v.\int_{\mathbb{R}^d}\frac{y_i}{|y|^{d+1}}f(x-y)dy = \lim_{\varepsilon\to 0}\int_{|y|>\varepsilon}\frac{y_i}{|y|^{d+1}}f(x-y)dy$$

for $j = 1, \cdots, d$, and from there it follows the development of the *Calderón-Zygmund theory of singular integrals*.

[Orthogonal Polynomials]

2.-Convergence of Orthogonal Polynomial Series

We want to study an analogous problem for the case of classical orthogonal polynomials in $\mathbb{R}$.
First remember that the classical orthogonal polynomials are three families.

Classical Orthogonal Polynomials on the Real Line

Let Q_n be a monic polynomial of degree n, such that

$$\int_\Delta Q_n(x)\,x^k\,d\mu(x) = 0, \quad \forall k = 0,1,2,\cdots,n-1$$

Polynomials	Q_n	Δ	$\mu(dx)$
Jacobi	$P_n^{\alpha,\beta}$	$[-1,1]$	$\mu_{\alpha,\beta}(dx) = (1-x)^\alpha(1+x)^\beta\,dx, \quad \alpha,\beta > -1.$
Gegenbauer	C_n^α	$[-1,1]$	$\mu_\alpha(dx) = (1-x^2)^\alpha\,dx, \quad \alpha = \beta > -1.$
Legendre	P_n	$[-1,1]$	$dx, \quad \alpha = \beta = 0.$
Chebyshev I	T_n	$[-1,1]$	$\mu_{-1/2}(dx) = (1-x^2)^{-\frac{1}{2}}\,dx, \quad \alpha = \beta = -\frac{1}{2}.$
Chebyshev II	U_n	$[-1,1]$	$\mu_{1/2}(dx) = (1-x^2)^{\frac{1}{2}}\,dx, \quad \alpha = \beta = \frac{1}{2}.$
Laguerre	L_n^α	$[0,\infty)$	$\lambda_\alpha(dx) = x^\alpha e^{-x}\,dx, \quad \alpha > -1.$
Hermite	H_n	$\mathbb{R}$	$\gamma(dx) = e^{-x^2}\,dx\,.$

The main references are:

- G. Szegő, **Orthogonal Polynomials**, Amer. Math. Soc., 1975.
- T. S. Chihara, **An Introduction to Orthogonal Polynomials**, Gordon and Breach, 1978.

- There are good news and bad news about the convergence of the orthogonal expansions for classical polynomials.
- First of all, the convergence in L^2-norm is true for the three families and the proof is totally analogous to the one given for the trigonometric system; we only need to prove that those families are complete in the corresponding L^2 space.
- Thus, we have that in order to prove the L^2 convergence of corresponding orthogonal expansions we have:
  - Needs to be proven that the orthogonal family $\{P_n^{\alpha,\beta}\}$ is complete on $L^2([-1,1],\mu_{\alpha,\beta})$, see Theorem 3.1.5 of Szégo's book.
  - Needs to be proven that the orthogonal family $\{L_n^{\alpha}\}$ is complete on $L^2([0,\infty),\lambda_\alpha)$, see Theorem 5.7.1 of Szégo's book.
  - Needs to be proven that the orthogonal family $\{H_n\}$ is complete on $L^2((-\infty,\infty),\gamma)$, see Theorem 5.7.1 of Szégo's book, also Pollard mentions that it can be obtained because the moment problem

    $$\mu_n = \int_{-\infty}^{\infty} x^n \gamma(dx)$$

    is determined.

- The analysis of the convergence of Fourier-Jacobi series in L^p has a long history. The first result was obtained by H. Pollard in a series of papers from 1946 to 1949 where he studied the case of the Legendre polynomials, then the case of the Gegenbauer polynomials and finally the case of the Jacobi polynomials,

Definition 3

For $1 \leq p < \infty$ consider the Banach space $L^p([-1,1], \mu_{\alpha,\beta})$ of p-th power integrable functions with respect to the *Jacobi measure* on $[-1,1]$, $\mu_{\alpha,\beta}(dx) = (1-x)^{\alpha}(1+x)^{\beta}dx$, $\alpha, \beta > -1$, with norm

$$\|f\|_{p,\mu_{(\alpha,\beta)}} = \left(\int_{-1}^{1} |f(x)|^p \mu_{\alpha,\beta}(dx) \right)^{\frac{1}{p}}$$

The family $\{P_n^{\alpha,\beta}\}$ is the orthogonal family with respect to the inner product

$$\langle f, g \rangle_{\alpha,\beta} = \int_{-1}^{1} f(x)\, g(x) \mu_{\alpha,\beta}(dx).$$

More precisely,

$$\int_{-\infty}^{\infty} P_n^{(\alpha,\beta)}(y) P_m^{(\alpha,\beta)}(y)\, \mu_{\alpha,\beta}(dx) = h_n^{(\alpha,\beta)} \delta_{n,m},$$

for $n, m = 0, 1, 2, \cdots$

where

$$h_n^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}}{(2n+\alpha+\beta+1)} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}.$$

The Jacobi polynomials $\{P_n^{\alpha,\beta}\}$ are also polynomial solutions of the *Jacobi differential equation*,

$$\left(1-x^2\right)y'' + [\beta-\alpha-(\alpha+\beta+2)x]\,y' + \gamma_n y = 0,$$

where, $\gamma_n = n(n+\alpha+\beta+1)$, for $n = 0,1,2,\ldots$.
Thus, $P_n^{(\alpha,\beta)}$ is an eigenfunction of the (one-dimensional) second order diffusion operator

$$\mathscr{L}^{\alpha,\beta} = -(1-x^2)\frac{d^2}{dx^2} - (\beta-\alpha-(\alpha+\beta+2)x)\frac{d}{dx},$$

associated with the eigenvalue $\lambda_n^{\alpha+\beta} = n(n+\alpha+\beta+1)$. $\mathscr{L}^{\alpha,\beta}$ is called the *Jacobi differential operator.*

Definition 4

Given $f \in L^1[-1,1](\mu_{\alpha,\beta})$ the *Fourier-Jacobi coefficients* of f are defined as

$$\hat{f}_J(n) = \{h_n^{(\alpha,\beta)}\}^{-1} \int_{-1}^{1} f(y) P_n^{(\alpha,\beta)}(y)\,\mu_{\alpha,\beta}(dy)$$

and define the partial sum operators of f as

$$s_n^J(f,x) = \sum_{k=0}^{n} \hat{f}_J(k) P_k^{(\alpha,\beta)}(x)$$

- Consider the *Dirichlet-Szẽgo kernel* for the Jacobi polynomials,

$$K_n^{(\alpha,\beta)}(x,y) = \sum_{\nu=0}^{n} \left\{h_\nu^{(\alpha,\beta)}\right\}^{-1} P_n^{(\alpha,\beta)}(x)\, P_n^{(\alpha,\beta)}(y),$$

- Using the *recurrent relation* for $\{P_n^{(\alpha,\beta)}\}$ we get the *Christofell-Darboux* formula,

$$K_n^{(\alpha,\beta)}(x,y) = \frac{2^{-\alpha-\beta}}{2n+\alpha+\beta+2} \frac{\Gamma(n+2)\Gamma(n+\alpha+\beta+2)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)} \frac{P_{n+1}^{(\alpha,\beta)}(x)\, P_n^{(\alpha,\beta)}(y) - P_n^{(\alpha,\beta)}(x)\, P_{n+1}^{(\alpha,\beta)}(y)}{x-y}.$$

For reasons that will be clear soon $K_n^{(\alpha,\beta)}(x,y)$ is called the *Dirichlet-Szẽgo kernel*.

- Then, the partial sums $S_n(f)$ can be represented as

$$\begin{aligned} s_n^J(f,x) &= \sum_{k=0}^{n} \hat{f}_J(k) P_k^{(\alpha,\beta)}(x) \\ &= \sum_{k=0}^{n} \{h_k^{(\alpha,\beta)}\}^{-1} \int_{-1}^{1} f(y) P_k^{(\alpha,\beta)}(y) \mu_{\alpha,\beta}(dy) P_k^{(\alpha,\beta)}(x) \\ &= \int_{-1}^{1} f(y) \left[\sum_{k=0}^{n} \{h_k^{(\alpha,\beta)}\}^{-1} P_k^{(\alpha,\beta)}(y) P_k^{(\alpha,\beta)}(x)\right] \mu_{\alpha,\beta}(dy) \end{aligned}$$

- Thus using the Christofell-Darboux formula, we have

$$\begin{aligned} s_n^J(f,x) &= \frac{2^{-\alpha-\beta}}{2n+\alpha+\beta+2}\frac{\Gamma(n+2)\Gamma(n+\alpha+\beta+2)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)} \\ &\qquad \times \int_{-1}^{1} f(y)\frac{P_{n+1}^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(y)-P_n^{(\alpha,\beta)}(x)P_{n+1}^{(\alpha,\beta)}(y)}{x-y}\mu^{\alpha,\beta}(dy) \\ &= \int_{-1}^{1} f(y)K_n^{(\alpha,\beta)}(x,y)\mu^{\alpha,\beta}(dy). \end{aligned}$$

- Observe that the Dirichlet-Széğo kernel is a singular kernel, closed related to the Hilbert transform already mentioned in the previous section.
- The main result obtained by H. Pollard is the following

Theorem 2 (H. Pollard)

Let $\alpha,\beta \geq -1/2$,

$$m(\alpha,\beta) = 4\,Max\left\{\frac{\alpha+1}{2\alpha+3},\frac{\beta+1}{2\beta+3}\right\},$$

and

$$M(\alpha,\beta) = 4\,min\left\{\frac{\alpha+1}{2\alpha+1},\frac{\beta+1}{2\beta+1}\right\}.$$

Then, for any $f \in L^p(\mu^{\alpha,\beta})$ with $m(\alpha,\beta) < p < M(\alpha,\beta)$, the Fourier-Jacobi expansion of f converges to f in $L^p(\mu^{\alpha,\beta})$-norm i.e. $\lim_{n\to\infty} \|f - s_n^H(f,\cdot)\|_{p,\mu_{(\alpha,\beta)}} = \lim_{n\to\infty}\left(\int_{-1}^{1}\left|f(x)-s_n^J(f,x)\right|^p \mu^{\alpha,\beta}(dx)\right)^{\frac{1}{p}} = 0.$

The Jacobi case

- In order to prove this result using the L^p continuity of the Hilbert transform, Pollard decomposed the Dirichlet-Szĕgo kernel $K_n^{(\alpha,\beta)}$ as

$$K_n^{(\alpha,\beta)}(x,y) = \alpha_n T_1(n,x,y) + \beta_n \left[T_2(n,x,y) + T_3(n,y,x)\right],$$

where

$$\begin{aligned} T_1(n,x,y) &= (n+1)\,P_n^{(\alpha,\beta)}(x)\,P_n^{(\alpha,\beta)}(y), \\ T_2(n,x,y) &= \frac{n\left(1-y^2\right)P_n^{(\alpha,\beta)}(x)\,P_{n-1}^{(\alpha+1,\beta+1)}(y)}{x-y} \\ T_3(n,x,y) &= \frac{n\left(1-x^2\right)P_n^{(\alpha,\beta)}(y)\,P_{n-1}^{(\alpha+1,\beta+1)}(x)}{y-x} \\ &= T_2(n,y,x), \end{aligned}$$

with coefficients

$$\alpha_n = \left\{h_n^{(\alpha,\beta)}\right\}^{-1}\frac{(n+\alpha+\beta+1)}{(n+1)(2n+\alpha+\beta+1)} = \frac{\Gamma(n+1)\Gamma(n+\alpha+\beta+2)}{2^{\alpha+\beta+1}\;(n+1)\Gamma(n+\alpha+1)\Gamma(n+\beta+1)},$$

$$\beta_n = \left\{h_n^{(\alpha,\beta)}\right\}^{-1}\frac{(n+\alpha+\beta+1)}{2n(2n+\alpha+\beta+1)} = \frac{\Gamma(n)\Gamma(n+\alpha+\beta+2)}{2^{\alpha+\beta+1}\;\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}.$$

These constants $|\alpha_n|$ and $|\beta_n|$ can be bounded above by a constant depending only α and β.

- This decomposition, which is now known as *Pollard decomposition*, is actually true for orthogonal polynomials of measures μ with compact support in $[-1,1]$.
- As before, by the the Banach-Steinhaus uniform boundedness principle the L^p convergence of the partial sums is equivalent to prove they are L^p bounded

$$\|s_n^J(f,\cdot)\|_{p,\mu_{(\alpha,\beta)}} \leq C\|f\|_{p,\mu_{(\alpha,\beta)}}.$$

- The main tools for boundedness of the operators defined by the kernels T_1, T_2 and T_3 are pointwise boundedness of the Jacobi polynomials,

$$\left|P_n^{(\alpha,\beta)}(x)\right| \leq Cn^{-1/2}\left(1-x+n^{-2}\right)^{(-1/2)\alpha-1/4},$$

where C, is a constant independent of x and n, and $0 \leq x \leq 1$ and the L^p continuity, **with respect to weights**, of the *Hilbert transform*
- This is one motivation of the A_p-*weight theory* in harmonic analysis initiated by B. Muckenhoupt and R. Wheeden in the early 70's for the Hilbert transform and the Hardy-Littlewood maximal function.
- Additionally, Pollard also proved that if $p < 4\,Max\left\{\frac{\alpha+1}{2\alpha+3}, \frac{\beta+1}{2\beta+3}\right\}$ or $p > 4\,min\left\{\frac{\alpha+1}{2\alpha+1}, \frac{\beta+1}{2\beta+1}\right\}$ there exists a function in $L^p(\mu^{\alpha,\beta})$ such that its Fourier-Jacobi series does not converge.
- In 1952 J. Newman, W. Rudin study the case of the extreme points $p = 4\,Max\left\{\frac{\alpha+1}{2\alpha+3}, \frac{\beta+1}{2\beta+3}\right\}$, $p = 4\,min\left\{\frac{\alpha+1}{2\alpha+1} \cdot \frac{\beta+1}{2\beta+1}\right\}$ proving that in that case the L^p convergence also fails.
- Observe that for the case $\alpha = \beta = -1/2$, which is the case of Chebyshev polynomials of the first type, the range obtained is $1 < p < \infty$ which follows from M. Riesz' classical theorem on Fourier series and the case $\alpha = \beta = 0$ that is the case of Legendre polynomials the range is $4/3 < p < 4$.

- Actually Pollard's approach in his paper is far more general, giving seven hypothesis, his famous (H-1) to (H-7) hypothesis for a weight $\omega(x)$ such that if satisfied for some $p > 1$ then, the corresponding Fourier series converges in L^p-norm. Additionally, Pollard introduced a general class of weights, that trivially includes the Jacobi (and therefore the Gegenbauer) weights. A weight ω is said to belong to the class $\mathscr{B}$ if

$$\omega(x) = t(x)(1-x)^{\alpha}(1+x)^{\beta},$$

where $\alpha, \beta \geq -1/2$ and $t(x)$ is positive, has continuous derivative and

$$t'(x+h) - t'(x) = o(\ln^{-2}|h|), \text{ as } h \to 0, \text{ uniformly in } [-1,1].$$

- For weights in class $\mathscr{B}$, S. Bernstein had proved basically that hypothesis (H-1) through (H4) are satisfied. Also, for $\omega \in \mathscr{B}$ it is easy to prove that hypothesis (H-5) and (H-6) provided

$$4\, Max\left\{\frac{\alpha+1}{2\alpha+3}, \frac{\beta+1}{2\beta+3}\right\} < p < 4\, min\left\{\frac{\alpha+1}{2\alpha+1} . \frac{\beta+1}{2\beta+1}\right\}.$$

Therefore, for $\omega \in \mathscr{B}$ it is enough to verify hypothesis (H-7): the operators $T_{\pm}$ defined as

$$T_{\pm}f(x) = \int_{-1}^{1} K_{\pm}(x,y) f(y)dy,$$

where $f \in L^p([-1,1])$ and the kernels $K_{\pm}$ are defined as

$$K_{\pm}(x,y) = \left|\frac{((1-y^2)/(1-x^2))^{\pm 1/4}(\omega(y)/w(x))^{1/2-1/p} - 1}{x-y}\right|,$$

are bounded in $L^p([-1,1])$. Pollard's proof for the Gegenbauer and Jacobi cases follows that scheme.

- The outline given above is a direct adaptation of his arguments for the Jacobi case, that an undergraduate student of mine in Venezuela, Joel Magallanes, did under my supervision as *tesis de licenciatura* at Universidad Central de Venezuela in 2003.
- A little later in 1950, G.M. Wing extended Pollard's result for Fourier-Bessel expansions for $4/3 < p < 4$, for Jacobi orthogonal functions expansions, and more general, for orthogonal functions expansions associated to weights that satisfy Pollard's (H-1)-(H-4) hypothesis for $4/3 < p < 4$, for orthogonal functions expansions associated to weights $\omega \in \mathscr{B}$ for $4/3 < p < 4$, and for orthogonal polynomials expansions associated to weights $\omega \in \mathscr{B}$ for $4\, Max\left\{\frac{\alpha+1}{2\alpha+3}, \frac{\beta+1}{2\beta+3}\right\} < p < 4\, min\left\{\frac{\alpha+1}{2\alpha+1} \cdot \frac{\beta+1}{2\beta+1}\right\}$.
- In 1969 B. Muckenhoupt filled the gap extending Pollard's result for all $\alpha, \beta > -1$, instead of only $\alpha, \beta > -1/2$, and a general result still using Pollard's decomposition but for weights of the form $\omega_a, b(x) = (1-x)^{ap}(1+x)^{bp}$ for $a, b \in \mathbb{R}$ such that

$$|a + 1/p - (\alpha+1)/2| \leq \text{mín}\{1/4, (\alpha+1)/2\}\ |b + 1/p - (\beta+1)/2| \leq \text{mín}\{1/4, (\beta+1)/2\},$$

that includes the Jacobi weights, for the case $a = \alpha/p,\ b = \beta/p$, and using a weighted inequality for the Hilbert transform: there exists a constant C_p independent of f such that

$$\int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} \frac{f(y)}{x-y} |x|^r (1+|x|)^{s-r} dy \right|^p dx \leq C_p \int_{-\infty}^{\infty} \left| f(y)|y|^R (1+|y|)^{S-R} \right|^p dy,$$

for $1 < p < \infty, r > -1/p, s < 1-1/p, S > -1/p, R < 1-1/p, r \geq R$ and $s \leq S$.
- It is easy to see that these conditions, for the case $\alpha, \beta > -1/2$, give Pollard's conditions $m(\alpha,\beta) < p < M(\alpha,\beta)$.

The Jacobi case

- In 1991, J. J. (Chicho) Guadalupe, M Pérez, F. J. Ruiz and J. L. Varona complete the study of the L^p-boundedness of partial sums related to the generalized Jacobi weights. Additionally, they studied the generalized Jacobi weights with mass points on the interval $[-1,1]$. Also, J. L. Varona obtained some additional results in that direction.
- Finally, if consider the *Sobolev inner product*

$$\langle f,g\rangle_S = \int_a^b f(x)g(x)d\mu_1(x) + \lambda \int_a^b f'(x)g'(x)d\mu_2(x), \quad \lambda > 0$$

where μ_1, μ_2 are positive Borel measures with support (a,b).
- The theory of Sobolev orthogonal polynomials has been studied intensively in the last 30 years; we recommend the survey of F. Marcellán, Y. Xu and the references therein to get a good overview.
- For instance, we can consider the convergence of the Fourier series in terms of orthonormal polynomials associated with the inner Sobolev product where (μ_1, μ_2) forms a coherent pair of measure. Consider pairs where one of the measures is a Jacobi measure $\mu_{\alpha,\beta}$ It may happen two possibilities:
i) (μ_1, μ_2) is a coherent pair of measures of Jacobi type I if $\mu_2 = \mu_{\alpha,\beta}$ and depending on α, μ_1 would be
  a) If $\alpha > 0, d\mu_1(x) = (\xi - x)(1-x)^{\alpha-1}(1+x)^{\beta-1}dx$ with $\xi \geq 1$.
  b) If $\alpha = 0, d\mu_1(x) = (1+x)^{\beta-1}dx + M\delta(1)$, with $M \geq 0$
  c) If $-1 < \alpha < 0, d\mu_1(x) = (1-x)^{\alpha}(1+x)^{\beta-1}dx$.

ii) (μ_1, μ_2) is a coherent pair of measures of Jacobi type II if $\mu_1 = \mu_{\alpha,\beta-1}$ and

$$d\mu_2(x) = \frac{1}{\xi - x}(1-x)^{\alpha+1}(1+x)^{\beta}dx + M\delta(\xi), \quad \xi \geq 1, M \geq 0$$

- Let (μ_1, μ_2) be a coherent pair of measures of Jacobi type, we define the space $W^p_{1,2}$, for $1 \leq p < \infty$, as the space of measurable functions f defined on [-1,1] such that there exists f' almost everywhere and

$$\|f\|^p_{W^p_{1,2}} = \|f\|^p_{L^p(\mu_1)} + \lambda \left\|f'\right\|^p_{L^p(\mu_2)} < \infty$$

Let $\{R_n\}_n$ be the sequence of orthonormal polynomials with respect to the Sobolev inner product defined above. Let $s^R_n(f)$ be the n-th partial sum given by

$$s^R_n(f,x) = \sum_{k=0}^{n} \hat{f}_R(k) R_k(x), \quad \hat{f}_R(k) = \langle R_k, f\rangle_S$$

- O. Ciaurri & M. Minguéz obtained in 2020 the following results.

Theorem 3

Let $\alpha > -1, \beta > 0$ and $1 < p < \infty$. Let (μ_1, μ_2) be a coherent pair of measures of Jacobi type I. Then

$$\left\|s^R_n(f)\right\|_{W^p_{1,2}} \leq C\|f\|_{W^p_{1,2}}$$

with a constant C independent of n and f, if and only if the following conditions hold

$$\left|(\alpha+1)\left(\frac{1}{p} - \frac{1}{2}\right)\right| < \frac{1}{4}, \quad \left|(\beta+1)\left(\frac{1}{p} - \frac{1}{2}\right)\right| < \frac{1}{4}.$$

Theorem 4

Let $\alpha > -1, \beta > 0$ and $1 < p < \infty$. Let (μ_1, μ_2) be a coherent pair of measures of Jacobi type II where $M = 0$. Then

$$\left\| s_n^R(f) \right\|_{W^p_{1,2}} \leq C \|f\|_{W^p_{1,2}}$$

with a constant C independent of n and f, if and only if the following conditions hold

$$\left| (\alpha + 2) \left(\frac{1}{p} - \frac{1}{2} \right) \right| < \frac{1}{4}, \quad \left| (\beta + 1) \left(\frac{1}{p} - \frac{1}{2} \right) \right| < \frac{1}{4}.$$

If the space $W_{1,2}$ was complete and the polynomials formed a dense class, again the the Banach-Steinhaus the boundedness of $s_n^R(f)$ is equivalent to the convergence in $W^p_{1,2}$ and that happen if $\alpha > -1, \beta > 0$ and (μ_1, μ_2) is a coherent pair of Jacobi measures with $M = 0$ for the type II.

The Hermite case

" If in $\left(1-x^2\right)^{\lambda-1/2}$ we replace x by $x\lambda^{-1/2}$, and take the limit as $\lambda \to \infty$, the weight function becomes e^{-x^2}, and the corresponding polynomials those of Hermite on $(-\infty, \infty)$. The inequality (5) suggests that in this case p-mean convergence holds only for $p = 2$. This, and a similar result for Laguerre series, will be confirmed by suitable counterexamples."

THE MEAN CONVERGENCE OF ORTHOGONAL SERIES. II
Harry Pollard 1948

Definition 5

For $1 \leq p < \infty$ consider the Banach space $L^p(\mathbb{R}, \gamma)$ of p-th power integrable functions with respect to $\gamma_1(dx) = \frac{1}{\sqrt{\pi}} e^{-x^2} dx$,, be the *Gaussian measure* on $\mathbb{R}$, with norm

$$\|f\|_{p,\gamma} = \left(\int_{-\infty}^{\infty} |f(x)|^p \gamma(dx) \right)^{\frac{1}{p}}$$

The family $\{H_n\}$ is the orthogonal family with respect to the inner product

$$\langle f, g \rangle_\gamma = \int_{-\infty}^{\infty} f(x)\, g(x) \gamma(dx).$$

More precisely,

$$\int_{-\infty}^{\infty} H_n(y) H_m(y)\, \gamma(dx) = 2^n n! \delta_{n,m},$$

for $n, m = 0, 1, 2, \cdots$

The Hermite polynomials $\{H_n\}$ are a polynomial solutions of the *Hermite equation* with parameter n,

$$LH_n(x) = \frac{1}{2}\frac{d^2}{dx}H_n(x) - x\frac{d}{dx}H_n(x) + nH_n(x) = 0,$$

or equivalently H_n is an eigenfunction of the one dimensional *Ornstein-Uhlenbeck operator*

$$L = \frac{1}{2}\frac{d^2}{dx} - x\frac{d}{dx}$$

with eigenvalue $-n$.

Definition 6

Given $f \in L^1(\mathbb{R}, \gamma)$ the n-th *Fourier-Hermite coefficients* of f are defined as

$$\hat{f}_H(n) = \frac{1}{2^n n!}\int_{-\infty}^{\infty} f(y)H_n(y)\gamma(dy)$$

and define the partial sum operators of f as

$$s_n^H(f,x) = \sum_{k=0}^{n} \hat{f}_H(k)H_n(x)$$

By a standard argument, one can get an integral representation for the partial sums

$$s_n^H f(x) = \sum_{k=0}^{n} \frac{1}{2^k k!}\int_{-\infty}^{\infty} f(y)H_k(y)\gamma(dy)H_k(x) = \int_{-\infty}^{\infty}\left[\sum_{k=0}^{n}\frac{H_k(x)H_k(y)}{2^k k!}\right]f(y)\,\gamma(dy) = \int_{-\infty}^{\infty} D_n^H(x,y)f(y)\,\gamma(dy),$$

where $D_n^H(x,y)$ is called the *Hermite Dirichlet-Szegő's kernel*.

The Hermite case

By the *Christofell-Darboux's formula*, we get the following representation of $D_n^H(x,y)$

$$D_n^H(x,y) = \sum_{k=0}^{n} \frac{H_k(x)H_k(y)}{2^k k!} = \left(\frac{1}{2^{n+1} n!}\right) \frac{H_{n+1}(x)H_n(y) - H_n(x)H_{n+1}(y)}{x-y}.$$

- In spite that at first this may look similar (actually easier!) than the Jacobi case and the fact that it can be proved that polynomials are dense in $L^p(\gamma)$ for any $1 \leq p < \infty$, H. Pollard proved that $s_n^H f \to f$ in $L^p(\gamma_1)$, that is

$$\int_{-\infty}^{\infty} |s_n^H f(x) - f(x)|^p \gamma_1(dx) \to 0,$$

as $n \to \infty$, **if and only if** $p = 2$ using the fact that the Hermite polynomials are a limiting case of the Gegenbauer polynomials, but $p = 2$.

- The motivation follows from Pollard's condition on Theorem 2. Remember that the *Gegenbauer polynomials* $\{C_n^\lambda\}$ are (using Szĕgo normalization) defined as

$$C_n^\lambda(x) = \frac{(2\lambda)_n}{(\lambda+1/2)_n} P_n^{(\lambda-1/2,\lambda-1/2)}(x),$$

- They have the following asymptotic relation for the Hermite polynomials and the Gegenbauer polynomials

$$\lim_{\lambda\to\infty} \lambda^{-n/2} C_n^\lambda(x\lambda^{-1/2}) = \frac{H_n(x)}{n!}.$$

- Now, for the case of Gegenbauer case, Pollard's condition becomes

$$2-\frac{1}{\lambda+1}=4\left(\frac{\lambda+1/2}{2\lambda+2}\right)<p<4\left(\frac{\lambda+1/2}{2\lambda}\right)=2+\frac{1}{\lambda}$$

and both ends clearly tends to 2 as $\lambda\to\infty$.

- Pollard not only gave this motivation but he also gave an explicit counterexample! Pollard's counterexample is the following: given $1<p<2$, let us consider the function

$$f(x)=e^{cx^2},$$

with $\frac{1}{2}<c<\frac{1}{p}$. Then $f\in L^p(\gamma)$. It can be shown that for any $k\in\mathbb{N}$,

$$\widehat{f}_H(2k+1)=0 \quad \text{and} \quad \widehat{f}_H(2k)=M\left(\frac{c}{1-c}\right)^k\frac{1}{4^k k!}.$$

We have, using a refined estimate from Szĕgo's book (8.22.8), there is a constant $A>0$ such that

$$\begin{aligned}\left(\int_{-\infty}^{\infty}e^{-x^2}\left|H_{2n}(x)\right|^p dx\right)^{1/p} &\geqq A\int_{-\infty}^{\infty}e^{-x^2}\left|H_{2n}(x)\right|dx\\ &\geqq A\int_{\pi}^{2\pi}e^{-x^2}\left|H_{2n}(x)\right|dx\\ &\geqq A\frac{(2n)!}{n!}\int_{\pi}^{2\pi}e^{-x^2/2}\left|\cos\left\{(2n+1)^{1/2}x\right\}\right|dx\end{aligned}$$

- Now let $k \to \infty$ through those values of k for which $(2k+1)^{1/2}$ is an integer N_k. Then

$$\begin{aligned} \operatorname*{lím\,sup}_{k\to\infty} \hat{f}_H(2k) \left(\int_{-\infty}^{\infty} |H_{2k}(x)|^p e^{-x^2} dx \right)^{1/p} &\geq \operatorname*{lím\,sup}_{k\to\infty} \frac{A}{N_k} \left(\frac{c}{1-c} \right)^{N_k} \int_{N_k\pi}^{2N_k\pi} |\cos y| dy \\ &\geq \operatorname*{lím\,sup}_{k\to\infty} A \left(\frac{c}{1-c} \right)^{N_k} = \infty, \end{aligned}$$

since $c > 1/2$.

- Since again by the the Banach-Steinhaus uniform boundedness principle the L^p convergence of the partial sums is equivalent to prove they are L^p bounded,

$$\|s_n^H(f,\cdot)\|_{p,\gamma} \leq C\|f\|_{p,\gamma},$$

the computation above proves that $\{s_n^H(f)\}$ are **not** bounded for $f(x) = e^{cx^2}$.

Definition 7

For $1 \leq p < \infty$ consider the Banach space $L^p([0,\infty),\lambda_\alpha), \alpha > -1$ of p-th power integrable functions with respect to

$$\lambda_\alpha(dx) = x^\alpha e^{-x}\, dx,$$

be the *Gamma measure* on $[0,\infty)$, with norm

$$\|f\|_{p,\gamma} = \left(\int_0^\infty |f(x)|^p \lambda_\alpha(dx) \right)^{\frac{1}{p}}$$

The family $\{L_n^\alpha\}$ is the orthogonal family with respect to the inner product

$$\langle f, g \rangle_\alpha = \int_0^\infty f(x)\, g(x) \lambda_\alpha(dx).$$

More precisely,

$$\int_0^\infty L_n^\alpha(y) L_m^\alpha(y)\, \lambda_\alpha(dx) = \Gamma(\alpha+1) \binom{n+\alpha}{n} \delta_{n,m},$$

for $n, m = 0, 1, 2, \cdots$
The Laguerre polynomials $\{L_n^\alpha\}$ are a polynomial solutions of the *Laguerre equation* with parameter α, n,

$$x(L_n^\alpha(x))'' + (\alpha+1-x)(L_n^\alpha(x))' + nL_n^\alpha(x) = 0,$$

or equivalently L_n^α is an eigenfunction, corresponding to the eigenvalue $-n$ of the one dimensional *Laguerre operator*

$$\mathscr{L}^\alpha = x\frac{d^2}{dx} + (\alpha+1-x)\frac{d}{dx}.$$

Definition 8

Given $f \in L^1((0,\infty,\lambda_\alpha)$ the n-th *Fourier-Laguerre coefficients* of f are defined as

$$\hat{f}_L(n) = \frac{1}{\Gamma(\alpha+1)\binom{n+\alpha}{n}} \int_0^\infty f(y) L_n^\alpha(y) \gamma(dy)$$

and define the partial sum operators of f as

$$s_n^L(f,x) = \sum_{k=0}^{n} \hat{f}_L(k) L_n^\alpha(x).$$

Again, by a standard argument, one can get an integral representation for the partial sums

$$\begin{aligned} s_n^L f(x) &= \sum_{k=0}^{n} \frac{1}{\Gamma(\alpha+1)\binom{n+\alpha}{n}} \int_0^\infty f(y) L_k^\alpha(y) \lambda_\alpha(dy) L_k^\alpha(x) \\ &= \int_0^\infty \left[\sum_{k=0}^{n} \frac{L_k^\alpha(x) L_k^\alpha(y)}{\Gamma(\alpha+1)\binom{n+\alpha}{n}} \right] f(y)\, \lambda_\alpha(dy) = \int_0^\infty D_n^L(x,y) f(y)\, \lambda_\alpha(dy), \end{aligned}$$

where $D_n^L(x,y)$ is called the *Laguerre Dirichlet-Szegő's kernel.* By the *Christofell-Darboux's formula*, we get the following representation of $D_n^L(x,y)$

$$D_n^H(x,y) = \sum_{k=0}^{n} \frac{L_k^\alpha(x) L_k^\alpha(y)}{\Gamma(\alpha+1)\binom{n+\alpha}{n}} = \frac{n+1}{\Gamma(\alpha+1)\binom{n+\alpha}{n}} \frac{L_{n+1}^\alpha(x) L_n^\alpha(y) - L_n^\alpha(x) L_{n+1}^\alpha(y)}{x-y}.$$

- Since there is a close relationship between Hermite and Laguerre polynomials, since we have

$$\begin{aligned} H_{2n}(x) &= (-1)^n 2^{2n} n! L_n^{-1/2}(x^2) \\ H_{2n+1}(x) &= (-1)^n 2^{2n+1} n! x L_n^{1/2}(x^2). \end{aligned}$$

Thus we expect the same anomalous behavior with respect to the convergence of Laguerre orthogonal expansions in L^p-norm.

- Moreover, given the asymptotic relation between the Laguerre and Jacobi polynomials

$$\lim_{\beta\to\infty} P_n^{(\alpha,\beta)}(1-2x/\beta) = L_n^{\alpha}(x),$$

taking $\beta \to \infty$ in Pollard's condition of Theorem 2 we see that

$$m(\alpha,\beta) = 4\, Max\left\{\frac{\alpha+1}{2\alpha+3}, \frac{\beta+1}{2\beta+3}\right\} \to 4 Max\left\{\frac{\alpha+1}{2\alpha+3}, \frac{1}{2}\right\} = 2$$

and

$$M(\alpha,\beta) = 4\, min\left\{\frac{\alpha+1}{2\alpha+1}, \frac{\beta+1}{2\beta+1}\right\} \to 4\, min\left\{\frac{\alpha+1}{2\alpha+1}, \frac{1}{2}\right\} = 2$$

- Without details Pollard gave a function for the counterexample in this case, $f(x) = e^{cx},\ 1/2 < c < 1/p$.

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